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Abstract: Finding the exact, quantum corrected metric on the hypermultiplet moduli space in Type II string compactifications on Calabi-Yau threefolds is an outstanding open problem. We address this issue by relating the quaternionic-Kähler metric on the hypermultiplet moduli space to the complex contact geometry on its twistor space. In this framework, Euclidean D-brane instantons are captured by contact transformations between different patches. We derive those by recasting the previously known A-type D2-instanton corrections in the language of contact geometry, covariantizing the result under electro-magnetic duality, and using mirror symmetry. As a result, we are able to express the effects of all D-instantons in Type II compactifications concisely as a sum of dilogarithm functions. We conclude with some comments on the relation to microscopic degeneracies of four-dimensional BPS black holes and to the wall-crossing formula of Kontsevich and Soibelman, and on the form of the yet unknown NS5-brane instanton contributions.

Keywords: Differential and Algebraic Geometry, Topological Strings, Black Holes in String Theory, String Duality

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Contents

1 Introduction

Understanding non-perturbative corrections to the moduli space of hypermultiplets in $\mathcal{N} = 2$ supersymmetric string vacua in $D = 4$ and $D = 3$ dimensions is an outstanding open problem with a host of possible applications. Firstly, it would provide new checks of heterotic/Type II string duality, which has mainly been tested in the vector multiplet sector [1]. Secondly, it may yield new insights on geometric invariants of Calabi-Yau (CY) two- or threefolds, governing the contributions of Euclidean D-brane and NS5-brane instantons in Type II strings, M2 and M5-brane instantons in M-theory [2, 3], or worldsheet instantons in heterotic strings on K_3 [4, 5]. Thirdly, it may provide a very useful packaging of the BPS black hole degeneracies in four dimensions, via their relation to $D = 3$ BPS instantons [6]. Finally, it may be of practical use for phenomenological model building, since the scalar potential in gauged supergravity typically depends on the metric on the hypermultiplet branch, see e.g. [7–11].

Contrary to the vector multiplet sector, where the relevant special Kähler (SK) metrics can be obtained from a holomorphic prepotential, a major difficulty in attacking this problem has been the lack of a convenient parametrization of the quaternionic-Kähler (QK) metrics on the hypermultiplet moduli space. Recently, it has become clear that twistor techniques [12–15] are a powerful and practical tool for addressing this problem. The relation of these mathematical constructions to the projective superspace techniques developed in the physics literature in the context of $\mathcal{N} = 2$ supersymmetric sigma models was gradually understood in a series of works [16–21].

In particular, via Swann's construction [15] and the superconformal quotient construction [18], QK (non-Kähler) manifolds $\mathcal{M}$ in $4d$ real dimensions are locally in one-to-one correspondence with $4d + 4$ dimensional hyperkähler cones (HKC) $\mathcal{S}$, i.e. HK manifolds with an isometric SU(2) action and a homothetic Killing vector. The HK metric on $\mathcal{S}$ can be obtained from the complex symplectic structure on its twistor space $\mathcal{Z}_{\mathcal{S}}$, which in turn may be encoded in complex symplectomorphisms relating different locally flat patches [19]. When $\mathcal{S}$ is a HKC, the complex symplectic structure on $\mathcal{Z}_{\mathcal{S}}$ is homogeneous and descends to a complex contact structure on the twistor space $\mathcal{Z}$ of $\mathcal{M}$ [14]. The latter may be described by complex contact transformations across different locally flat patches [21]. Thus, the metric of a QK manifold can be encoded in a family of holomorphic functions on $\mathcal{Z}$ subject to consistency relations, reality conditions and gauge equivalence. This allows to by-pass the HKC $\mathcal{S}$ and its twistor space $\mathcal{Z}_{\mathcal{S}}$ altogether, even though the connection to projective superspace is most obvious from the viewpoint of $\mathcal{Z}_{\mathcal{S}}$.

In the absence of isometries, computing the actual QK metric on $\mathcal{M}$ (or HKC metric on $\mathcal{S}$) is in general difficult, as it requires determining the real (contact) twistor lines. It is however amenable to systematic approximation schemes when $\mathcal{M}$ is a small deformation of a well understood QK manifold. In particular, in [19, 21] we have given a general formalism for linear perturbations of toric[1] HK and QK manifolds.

The projective superspace description of the hypermultiplet moduli space at tree level in Type II compactifications has been worked out in [23, 24] (see also [25, 26] for some prescient work), and the one-loop correction was incorporated in [27, 28], generalizing earlier results [29–31]. Arguments for the absence of perturbative corrections beyond one-loop were given in [27, 30, 31]. More recently, by studying the fate of the worldsheet instanton corrections under S-duality, the authors of [32] were able to compute the D(-1) and D1-instanton corrections to the hypermultiplet metric in Type IIB string theory compactified on a CY threefold Y. Under mirror symmetry, these corrections translate into Euclidean D2-brane instantons (or M2-branes in M-theory [2]) wrapped on special Lagrangian submanifolds of the mirror CY threefold X [33, 34], recovering in particular

[1] A $4d$-dimensional HK manifold is toric if it has d commuting tri-holomorphic isometries. A $4d$-dimensional QK manifold is toric if it has $d + 1$ commuting isometries. Both of these cases are covered by the Legendre transform construction [16, 22].

the analysis of [35] in the conifold limit. However, these results do not include all D2-instanton contributions, since D(-1) and D1-instantons are mirror symmetric to D2-branes wrapping A-type cycles only, where A- and B-cycles refer to a symplectic basis of $H_3(X)$ adapted to the point of maximal unipotent monodromy (sometimes referred to as the large complex structure limit); neither do they include NS5-brane instantons (or M5-branes in M-theory).

The reason for restricting to A-type D2-instantons (or D(-1) and D1-instantons on the Type IIB side) is that standard projective superspace techniques rely on the existence of $d + 1$ commuting continuous isometries, which allow to dualize all hypermultiplets into tensor multiplets. Generic instanton contributions preserve only a discrete subgroup of the continuous isometries, and the resulting metric falls outside the class of metrics obtainable by the Legendre transform method [16, 22].[2] However, the general construction of HK (resp., QK) manifolds from complex symplectic (resp., contact) manifolds with a compatible real structure remains valid. It may well be feasible to determine the complex symplectic (or contact) structure on the twistor space exactly, e.g. by specifying a set of complex symplectomorphisms (or contact transformations) between different locally flat patches, even if the exact HK (QK) metric remains out of reach.

This strategy was applied recently to the case of $D = 3, \mathcal{N} = 4$ supersymmetric gauge theories in 2+1 dimensions, obtained from compactifying $D = 4, \mathcal{N} = 2$ supersymmetric gauge theories on a circle [38]. In this case, the moduli space is HK, and receives instanton corrections from 4D BPS solitons winding around the Euclidean circle. The elementary symplectomorphism induced by such a soliton can be computed unambiguously in field theory, and a natural way to combine contributions from mutually non-local solitons suggests itself [38]. While the (indexed) one-particle BPS spectrum jumps across lines of marginal stability (LMS), the multi-particle BPS spectrum and more generally any physical observable should be smooth across the LMS. Indeed, the authors of [38] show that the HK metric derived from the symplectic structure is regular across the LMS, provided the change in the one-particle BPS spectrum satisfies constraints identical in form to the wall-crossing formula for "generalized Donaldson-Thomas invariants" [39] (see [40] for a physics discussion of this formula).

In this paper, we initiate a similar study in the context of $\mathcal{N} = 2$ supergravity in four dimensions, and obtain the contributions of all A-type and B-type D-instantons (with vanishing NS5-brane charge) to the hypermultiplet metric in Type II compactifications. Following the roadmap laid out in [34], we proceed by covariantizing the known A-type contributions under electric-magnetic duality and using mirror symmetry (in the process, we clarify the action of the latter on the Ramond-Ramond (RR) potentials, at least in the large volume limit). We work in the "leading instanton approximation", treating the D-instantons as linear perturbations around the one-loop corrected geometry of [27][3] using

[2]HK and QK metrics obtainable from the generalized Legendre transform [36] have generically no isometries, but still possess a higher rank Killing tensor; the metric on the universal hypermultiplet in the presence of NS5-brane instantons has been argued to fall in this class [37].

[3]It is possible in principle to treat the A-type instantons exactly as in [32, 34, 41] and the B-type instantons as linear perturbations. While this constitutes a valid approximation in the limit of large complex structures or Kähler classes, this approach is not directly useful as it breaks electric-magnetic

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the formalism developed in [19, 21]. In particular, we obtain the instanton corrected twistor lines (4.13) and contact potential (4.17) (related to the Kähler potential on $\mathcal{Z}$ via (2.6)), and show that the D-instanton effects can be concisely summarized in a holomorphic function (4.10), expressed as a sum of dilogarithms, controlling the deformation of the complex contact structure on $\mathcal{Z}$.

Our results above should really be viewed as a parametrization of the instanton corrected hypermultiplet metric. While the coefficients of mixed A/B-type contributions are in principle new geometric invariants of the CY threefold, we do not know how to compute them, although they should be obtainable from the dual heterotic sigma model [1, 4, 5]. The analogy with the results of [39], most notably the appearance of the dilogarithm function, strongly suggests that these invariants should be identified with the generalized Donaldson-Thomas invariants defined in [39]. It is possible that using the wall-crossing formula of [39], possibly combined with some automorphy requirement, one may be able to fix these invariants completely. By reduction from 4D to 3D and T-duality on the circle, the same invariants should determine the exact micro-state degeneracies of 4D black holes, as we further discuss in section 5.1.

This paper is organized as follows. • In section 2, we summarize the twistorial description of general QK manifolds and the linear deformations of toric QK metrics. • In section 3 we describe the hypermultiplet moduli space at the perturbative level, and discuss the action of S-duality and mirror symmetry. • In section 4, we formulate the A-type instanton corrections in terms of the contact geometry on the twistor space, and use electric-magnetic duality and mirror symmetry to obtain the effect of mixed A and B-type instantons in the leading instanton approximation. We derive the instanton corrected twistor lines and Kähler potential, and suggest a construction of the instanton corrected twistor space beyond the leading instanton approximation in the spirit of [38]. • In section 5, we relate D-instanton corrections to the 4D hypermultiplet moduli space to corrections to the 3D vector multiplet moduli space induced by 4D BPS black holes, discuss the usefulness of this approach in incorporating the moduli dependence of the black hole micro-state degeneracies, comment on possible relations to the generalized Donaldson-Thomas invariants of [39] and on the form of NS5-instanton contributions. • In appendix A, we revisit the construction of the twistor space of the Ooguri-Vafa metric discussed in [38], and extend it to provide a rigorous construction of the twistor space of the hypermultiplet branch in the leading instanton approximation.

2 QK spaces, twistors and contact geometry

In this section, we give a streamlined summary of our recent work [21] on the twistor approach to QK geometry, retaining only the information relevant for the twistor space $\mathcal{Z}$ of $\mathcal{M}$, and with a few changes of notations in order to avoid cluttering.[4] Further

covariance.

[4] In particular, we drop the "hat" on all symbols $\hat{f}_{ij}, \hat{\mathcal{X}}^{[i]}, \hat{S}^{[ij]}, \hat{H}^{[ij]}$, replace the index $\flat$ by the subscript α, and rephrase all contact transformations in terms of the variable $\alpha^{[i]}$ rather than $\tilde{\xi}_\flat^{[i]}$. This removes the c_I dependence from the contact transformations (2.71) and (5.23) in [21]. Moreover, since only the anomalous dimensions in the patch $\mathcal{U}_\pm$ play a role in the present construction, we denote $c_\Lambda \equiv c_\Lambda^{[+]} = -c_\Lambda^{[-]}$, $c_\alpha \equiv c_\flat^{[+]} = -c_\flat^{[-]}$.

mathematical details can be found, e.g., in [14, 42].

2.1 General quaternionic-Kähler manifolds

A QK manifold $\mathcal{M}$ is a $4d$-dimensional Riemannian manifold whose holonomy is contained in $\mathrm{USp}(d) \times \mathrm{SU}(2)$. It admits a quaternionic structure, which locally yields three almost complex structures satisfying the algebra of the unit quaternions. $\mathcal{M}$ is conveniently described by its twistor space $\mathcal{Z}$, a $\mathbb{C}P^1$ bundle over $\mathcal{M}$, whose connection is given by the SU(2) part $\vec{p}$ of the Levi-Civita connection on $\mathcal{M}$. $\mathcal{Z}$ admits a canonical (integrable) complex structure and a Kähler-Einstein metric [14]. The latter can be written as

$$\mathrm{d}s^2_{\mathcal{Z}} = \frac{|D\mathbf{z}|^2}{(1+\mathbf{z}\bar{\mathbf{z}})^2} + \frac{\nu}{4}\,\mathrm{d}s^2_{\mathcal{M}}\,. \tag{2.1}$$

Here $\mathbf{z}$ is a complex coordinate[5] on $\mathbb{C}P^1$, $D\mathbf{z}$ is the canonical $(1,0)$ form

$$D\mathbf{z} \equiv \mathrm{d}\mathbf{z} + p_+ - \mathrm{i}p_3\,\mathbf{z} + p_-\,\mathbf{z}^2\,, \tag{2.2}$$

with $p_- = (p_+)^*$, $p_3 = (p_3)^*$ under complex conjugation, and $\nu = R/(4d(d+2))$ is a numerical constant which sets the constant curvature R of $\mathcal{M}$.

The kernel of $D\mathbf{z}$ endows $\mathcal{Z}$ with a complex contact structure [14, 42] (see e.g. [43] for a general introduction to contact geometry). The latter can be represented by a set of holomorphic one-forms $\mathcal{X}^{[i]}$ defined on an open covering $\hat{\mathcal{U}}_i$ of $\mathcal{Z}$, such that the holomorphic top form $\mathcal{X}^{[i]} \wedge (\mathrm{d}\mathcal{X}^{[i]})^d$ is nowhere vanishing. On each patch, $\mathcal{X}^{[i]}$ is proportional to $D\mathbf{z}$,

$$\mathcal{X}^{[i]} = 2\,e^{\Phi_{[i]}}\,\frac{D\mathbf{z}}{\mathbf{z}}\,, \tag{2.3}$$

where $\Phi_{[i]} \equiv \Phi_{[i]}(x^\mu, \mathbf{z})$ is a function on $\hat{\mathcal{U}}_i \subset \mathcal{Z}$ which we refer to as the "contact potential". It is holomorphic along the $\mathbb{C}P^1$ fiber, defined up to an additive holomorphic function on $\hat{\mathcal{U}}_i$, and chosen such that the right-hand side of (2.3) is a holomorphic (i.e. $\bar{\partial}$-closed) one-form. The reality constraint

$$\overline{\tau(\mathcal{X}^{[i]})} = -\mathcal{X}^{[\bar{\imath}]}\,, \tag{2.4}$$

where τ is the antipodal map acting as $\tau : \mathbf{z} \to -1/\bar{\mathbf{z}}$ on $\mathbb{C}P^1$ and relating the two patches $\hat{\mathcal{U}}_i$ and $\hat{\mathcal{U}}_{\bar{\imath}}$, requires that

$$\overline{\tau(\Phi_{[i]})} = \Phi_{[\bar{\imath}]}\,. \tag{2.5}$$

The real part of $\Phi_{[i]}$ provides a Kähler potential for the Kähler-Einstein metric on $\mathcal{Z}$ in the patch $\hat{\mathcal{U}}_i$,

$$K^{[i]}_{\mathcal{Z}} = \log\frac{1+\mathbf{z}\bar{\mathbf{z}}}{|\mathbf{z}|} + \mathrm{Re}\,\Phi_{[i]}(x^\mu, \mathbf{z})\,. \tag{2.6}$$

One way to compute the metric on $\mathcal{Z}$ and $\mathcal{M}$ would be to express (2.6) in terms of complex coordinates on $\mathcal{Z}$, which is in general difficult. Fortunately, we shall be able to obtain the metric without knowing this change of coordinates. Note that the metric (2.1) now rewrites as

$$\mathrm{d}s^2_{\mathcal{Z}} = \frac{1}{4}\left(e^{-2K_{\mathcal{Z}}}|\mathcal{X}|^2 + \nu\,\mathrm{d}s^2_{\mathcal{M}}\right)\,, \tag{2.7}$$

[5]Note, however, that the projection $\mathcal{Z} \to \mathbb{C}P^1$, $(u, \bar{u}) \mapsto \mathbf{z}$ is not holomorphic.

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consistently with [18] where (2.7) was written in a particular gauge.

By a simple extension of Darboux's theorem [42], one may choose the open covering $\hat{\mathcal{U}}_i$ such that on each patch $\mathcal{X}^{[i]}$ takes the canonical form

$$\mathcal{X}^{[i]} = \mathrm{d}\alpha^{[i]} + \xi^\Lambda_{[i]}\, \mathrm{d}\tilde{\xi}^{[i]}_\Lambda \,. \tag{2.8}$$

Here $\xi^\Lambda_{[i]}, \tilde{\xi}^{[i]}_\Lambda, \alpha^{[i]}$ $(\Lambda = 0, \dots, d-1)$ are local complex coordinates on $\mathcal{Z}$, smooth throughout the patch $\hat{\mathcal{U}}_i$. Moreover, these coordinates may be chosen to satisfy the reality conditions

$$\overline{\tau(\xi^\Lambda_{[i]})} = \xi^\Lambda_{[\bar{\imath}]} \,, \qquad \overline{\tau(\tilde{\xi}^{[i]}_\Lambda)} = -\tilde{\xi}^{[\bar{\imath}]}_\Lambda \,, \qquad \overline{\tau(\alpha^{[i]})} = -\alpha^{[\bar{\imath}]} \,. \tag{2.9}$$

While the form (2.8) can always be achieved locally by a choice of complex coordinates, the global complex contact structure on $\mathcal{Z}$ is encoded in the set of *complex contact transformations* which relate the two systems of complex coordinates $(\xi^\Lambda_{[i]}, \tilde{\xi}^{[i]}_\Lambda, \alpha^{[i]})$ and $(\xi^\Lambda_{[j]}, \tilde{\xi}^{[j]}_\Lambda, \alpha^{[j]})$ on the overlap $\hat{\mathcal{U}}_i \cap \hat{\mathcal{U}}_j$. Complex contact transformations are holomorphic transformations which obey

$$\mathcal{X}^{[i]} = f_{ij}^2\, \mathcal{X}^{[j]} \,, \tag{2.10}$$

for some nowhere vanishing holomorphic function f_{ij}^2 on $\hat{\mathcal{U}}_i \cap \hat{\mathcal{U}}_j$. They can generally be represented[6] by a holomorphic function $S^{[ij]}(\xi^\Lambda_{[i]}, \tilde{\xi}^{[j]}_\Lambda, \alpha^{[j]})$ of the "initial position" $\xi^\Lambda_{[i]}$, "final momentum" $\tilde{\xi}^{[j]}_\Lambda$ and "final action" $\alpha^{[j]}$ such that

$$\begin{aligned} \xi^\Lambda_{[j]} &= f_{ij}^{-2}\, \partial_{\tilde{\xi}^{[j]}_\Lambda} S^{[ij]} \,, & \tilde{\xi}^{[i]}_\Lambda &= \partial_{\xi^\Lambda_{[i]}} S^{[ij]} \,, \\ \alpha^{[i]} &= S^{[ij]} - \xi^\Lambda_{[i]} \partial_{\xi^\Lambda_{[i]}} S^{[ij]} \,, & f_{ij}^2 &= \partial_{\alpha^{[j]}} S^{[ij]} \,, \end{aligned} \tag{2.11}$$

on $\hat{\mathcal{U}}_i \cap \hat{\mathcal{U}}_j$. In particular, the contact potentials satisfy

$$e^{\Phi_{[i]}} = f_{ij}^2\, e^{\Phi_{[j]}} \,. \tag{2.12}$$

As explained in [19, 21], the functions $S^{[ij]}$ are subject to several conditions: (i) consistency conditions ensuring that the contact transformations compose properly on the triple overlap $\hat{\mathcal{U}}_i \cap \hat{\mathcal{U}}_j \cap \hat{\mathcal{U}}_k$,

$$S^{[ij]}(\xi^\Lambda_{[i]}, \tilde{\xi}^{[j]}_\Lambda, \alpha^{[j]}) = \left\langle S^{[ik]}(\xi^\Lambda_{[i]}, \tilde{\xi}^{[k]}_\Lambda, \alpha^{[k]}) - \lambda \left(\alpha^{[k]} + \xi^\Lambda_{[k]} \tilde{\xi}^{[k]}_\Lambda - S^{[kj]}(\xi^\Lambda_{[k]}, \tilde{\xi}^{[j]}_\Lambda, \alpha^{[j]}) \right) \right\rangle , \tag{2.13}$$

where the bracket denotes extremization over $\xi^\Lambda_{[k]}, \tilde{\xi}^{[k]}_\Lambda, \alpha^{[k]}$ and the Lagrange multiplier λ; (ii) gauge equivalence generated by holomorphic functions $T^{[i]}$ in each patch $\hat{\mathcal{U}}_i$,

$$\begin{aligned} S^{[ij]}(\xi^\Lambda_{[i]}, \tilde{\xi}^{[j]}_\Lambda, \alpha^{[j]}) \mapsto \Big\langle & T^{[i]}(\xi^\Lambda_{[i]}, \tilde{\xi}'^{[i]}_\Lambda, \alpha'^{[i]}) - \lambda_1 \left(\alpha'^{[i]} + \xi'^\Lambda_{[i]} \tilde{\xi}'^{[i]}_\Lambda - S^{[ij]}(\xi'^\Lambda_{[i]}, \tilde{\xi}'^{[j]}_\Lambda, \alpha'^{[j]}) \right) \\ & + \lambda_2 \left(\alpha^{[j]} + \xi^\Lambda_{[j]} \tilde{\xi}^{[j]}_\Lambda - T^{[j]}(\xi^\Lambda_{[j]}, \tilde{\xi}'^{[j]}_\Lambda, \alpha'^{[j]}) \right) \Big\rangle \,, \end{aligned} \tag{2.14}$$

[6] One way to see this is to "symplectize" the contact form, i.e. introduce an extra local complex variable $\nu^\alpha_{[i]}$ and consider the homogeneous symplectic form $\Omega^{[i]} = \mathrm{d}(\nu^\alpha_{[i]} \mathcal{X}^{[i]})$.

where the bracket denotes extremization over $\xi'^{\Lambda}_{[i]}, \tilde\xi'^{[i]}_{\Lambda}, \alpha'^{[i]}, \xi'^{\Lambda}_{[j]}, \tilde\xi'^{[j]}_{\Lambda}, \alpha'^{[j]}$ and the Lagrange multipliers λ_1, λ_2; and (iii) reality conditions ensuring (2.4),

$$\overline{\tau(S^{[ij]})} = -S^{[\bar{i}\bar{j}]} . \tag{2.15}$$

A particularly important case occurs when f^2_{ij} can be chosen all equal to one. In this case, $S^{[ij]}$ reduces to

$$S^{[ij]}(\xi^{\Lambda}_{[i]}, \tilde\xi^{[j]}_{\Lambda}, \alpha^{[j]}) = \alpha^{[j]} + S^{[ij]}(\xi^{\Lambda}_{[i]}, \tilde\xi^{[j]}_{\Lambda}) , \tag{2.16}$$

where $S^{[ij]}$ is the generating function of a symplectomorphism of the $(\xi^{\Lambda}, \tilde\xi_{\Lambda})$ phase space, and $\mathcal{X}$ becomes a global contact one-form with Reeb vector $\partial_{\alpha^{[i]}}$. The contact potentials $\Phi_{[i]}$ are then all equal to a single real function $\Phi(x^{\mu})$, constant along the $\mathbb{C}P^1$ fiber (as follows from (2.12) and the requirement of holomorphy in all patches). Toric QK manifolds discussed in section 2.2 below fall in this class, and so do hypermultiplet moduli spaces in the absence of NS5-brane instantons, as discussed in section 4.2.

In order to construct the QK metric on $\mathcal{M}$, one should first determine the "contact twistor lines", i.e. express the local complex coordinates $\xi^{\Lambda}_{[i]}, \tilde\xi^{[i]}_{\Lambda}, \alpha^{[i]}$ on $\mathcal{Z}$ in terms of the coordinates x^{μ} on the base and the coordinate $\mathbf{z}$ on the fiber. In the patch $\hat{\mathcal{U}}_+$ around $\mathbf{z} = 0$, the coordinates must be smooth up to specific singular terms compatible with the form of (2.3) [21],

$$\begin{aligned}
\xi^{\Lambda}_{[+]} &= \xi^{\Lambda,-1}_{[+]}\,\mathbf{z}^{-1} + \xi^{\Lambda,0}_{[+]} + \xi^{\Lambda,1}_{[+]}\,\mathbf{z} + \mathcal{O}(\mathbf{z}^2) , \\
\tilde\xi^{[+]}_{\Lambda} &= c_{\Lambda} \log \mathbf{z} + \tilde\xi^{[+]}_{\Lambda,0} + \tilde\xi^{[+]}_{\Lambda,1}\,\mathbf{z} + \mathcal{O}(\mathbf{z}^2) , \\
\alpha^{[+]} &= c_{\alpha} \log \mathbf{z} + c_{\Lambda}\, \xi^{\Lambda,-1}_{[+]}\,\mathbf{z}^{-1} + \alpha^{[+]}_0 + \alpha^{[+]}_1\,\mathbf{z} + \mathcal{O}(\mathbf{z}^2) , \\
\Phi_{[+]} &= \phi^0_{[+]} + \phi^1_{[+]}\,\mathbf{z} + \mathcal{O}(\mathbf{z}^2) .
\end{aligned} \tag{2.17}$$

Here the coefficients c_I, with the index I running over $\alpha, 0, \dots, d-1$, are complex numbers called "anomalous dimensions". As a result of the logarithmic singularity in (2.17), the last two reality conditions in (2.9) pick up additive constants, which however do not affect the reality condition on $\mathcal{X}^{[i]}$. Generically, all Laurent coefficients in (2.17) are determined from the lowest coefficients $\xi^{\Lambda,-1}_{[+]}, \tilde\xi^{[+]}_{\Lambda,0}, \alpha^{[+]}_0$ by imposing the gluing conditions (2.11), and parametrize the manifold $\mathcal{M}$, up to overall phase rotations of $\xi^{\Lambda,-1}_{[+]}$.

The quaternionic-Kähler metric g on $\mathcal{M}$ may then be recovered as follows (see [21] for more details and explicit examples). Firstly, by expanding $e^{-\Phi_{[+]}}\mathcal{X}^{[+]}$ in (2.8) around $\mathbf{z} = 0$, and comparing with (2.3), one may extract the SU(2) connection

$$\begin{aligned}
p_+ &= \frac{1}{2}\, e^{-\phi^0_{[+]}} \left(\xi^{\Lambda,-1}_{[+]} \mathrm{d}\tilde\xi^{[+]}_{\Lambda,0} + c_{\Lambda} \mathrm{d}\xi^{\Lambda,-1}_{[+]} \right) , \\
p_3 &= \frac{\mathrm{i}}{2}\, e^{-\phi^0_{[+]}} \left(\mathrm{d}\alpha^{[+]}_0 + \xi^{\Lambda,0}_{[+]} \mathrm{d}\tilde\xi^{[+]}_{\Lambda,0} + \xi^{\Lambda,-1}_{[+]} \mathrm{d}\tilde\xi^{[+]}_{\Lambda,1} \right) - \mathrm{i}\phi^1_{[+]} p_+ ,
\end{aligned} \tag{2.18}$$

and express the Laurent coefficients of the contact potential in terms of the Laurent coefficients of the contact twistor lines,

$$\begin{aligned}
e^{\phi^0_{[+]}} &= \frac{1}{2} \left(\xi^{\Lambda,-1}_{[+]} \tilde\xi^{[+]}_{\Lambda,1} + c_{\Lambda} \xi^{\Lambda,0}_{[+]} + c_{\alpha} \right) , \\
\phi^1_{[+]} &= \frac{1}{2}\, e^{-\phi^0_{[+]}} \left(\alpha^{[+]}_1 + 2\xi^{\Lambda,-1}_{[+]} \tilde\xi^{[+]}_{\Lambda,2} + \xi^{\Lambda,0}_{[+]} \tilde\xi^{[+]}_{\Lambda,1} + c_{\Lambda} \xi^{\Lambda,1}_{[+]} \right) .
\end{aligned} \tag{2.19}$$

Subsequently, one expands the holomorphic one-forms $\mathrm{d}\xi^\Lambda_{[+]}$, $\mathrm{d}\tilde\xi^{[+]}_\Lambda$ and $\mathrm{d}\alpha$ around $\mathbf{z}=0$ and projects them along the base $\mathcal{M}$, producing local one-forms on $\mathcal{M}$ of Dolbeault type $(1,0)$ with respect to the quaternionic structure J_3. A basis of these forms is given by

$$\Pi^a = \xi^{0,-1}_{[+]}\mathcal{V}^a - \xi^{a,-1}_{[+]}\mathcal{V}^0\,, \qquad \tilde\Pi_I = \xi^{0,-1}_{[+]}\tilde{\mathcal{V}}_I + c_I\mathcal{V}^0\,, \tag{2.20}$$

where a runs over $1,\dots,d-1$, and

$$\begin{aligned} &\mathcal{V}^\Lambda \equiv (\mathrm{d}-\mathrm{i}p_3)\xi^{\Lambda,-1}_{[+]}\,, \qquad \tilde{\mathcal{V}}_\Lambda \equiv \mathrm{d}\tilde\xi^{[+]}_{\Lambda,0} - \tilde\xi^{[+]}_{\Lambda,1}\,p_+ + \mathrm{i}c_\Lambda p_3\,, \\ &\tilde{\mathcal{V}}_\alpha \equiv \mathrm{d}\alpha^{[+]}_0 - c_\Lambda \mathrm{d}\xi^{\Lambda,0}_{[+]} - (\alpha^{[+]}_1 - c_\Lambda\xi^{\Lambda,1}_{[+]})p_+ + \mathrm{i}\,c_\alpha p_3\,, \end{aligned} \tag{2.21}$$

Then, one may compute the triplet of quaternionic 2-forms $\vec\omega$ from the curvature of the SU(2) connection

$$\mathrm{d}\vec p + \frac{1}{2}\,\vec p\times\vec p = \frac{\nu}{2}\,\vec\omega\,, \tag{2.22}$$

where ν is a constant related to the scalar curvature of $\mathcal{M}$. In particular, we have (without loss of generality we will set $\nu=2$ in the following)

$$\omega_3 = \mathrm{d}p_3 + 2\mathrm{i}p_+\wedge p_-\,, \tag{2.23}$$

and obtain the QK metric via $g=\omega_3\cdot J_3$.

2.2 Toric quaternionic-Kähler geometries

A particularly simple case occurs for toric QK manifolds, i.e. when the $4d$-dimensional QK manifold $\mathcal{M}$ admits $d+1$ commuting isometries. In this case, the moment maps associated to these isometries [44] provide $d+1$ independent global $\mathcal{O}(2)$ sections on the twistor space $\mathcal{Z}$ of $\mathcal{M}$, which can be taken to be the complex coordinates $\xi^\Lambda_{[i]}$ and the unit function. Thus, on all patches $\hat{\mathcal{U}}_i$, $\xi^\Lambda_{[i]}$ takes the form

$$\xi^\Lambda_{[i]} = \xi^\Lambda \equiv Y^\Lambda\mathbf{z}^{-1} + A^\Lambda - \bar Y^\Lambda\mathbf{z}\,. \tag{2.24}$$

Moreover, the U(1) action corresponding to phase rotations of $\mathbf{z}$ can be fixed by choosing $Y^0\equiv\mathcal{R}$ to be real. Supplemented by $d+1$ additional coordinates B_I to be defined below, $\mathcal{R}, Y^a, \bar Y^a, A^\Lambda$ provide a convenient coordinate system on $\mathcal{M}$.

On the overlap of two patches, the complex coordinates $\xi^\Lambda, \tilde\xi_\Lambda, \alpha$ must now be related by a complex contact transformation which preserves ξ^Λ and the unit function. This restricts the generating function $S^{[ij]}$ to the form

$$S^{[ij]} = \alpha^{[j]} + \xi^\Lambda_{[i]}\,\tilde\xi^{[j]}_\Lambda - H^{[ij]}(\xi^\Lambda_{[i]})\,, \tag{2.25}$$

where $H^{[ij]}(\xi^\Lambda)$ is a holomorphic function on $\hat{\mathcal{U}}_i\cap\hat{\mathcal{U}}_j$. The contact transformations (2.11) become

$$\tilde\xi^{[i]}_\Lambda = \tilde\xi^{[j]}_\Lambda - \partial_{\xi^\Lambda}H^{[ij]}\,, \qquad \alpha^{[i]} = \alpha^{[j]} - H^{[ij]} + \xi^\Lambda\partial_{\xi^\Lambda}H^{[ij]}\,, \tag{2.26}$$

and the transition function f^2_{ij} is now equal to one. The consistency conditions (2.13), gauge equivalence (2.14) and reality conditions (2.15) translate into

$$H^{[ij]} + H^{[jk]} = H^{[ik]}\,, \qquad H^{[ij]} \mapsto H^{[ij]} + T^{[i]} - T^{[j]}\,, \qquad \overline{\tau(H^{[ij]})} = -H^{[\bar{\imath}\bar{\jmath}]}\,. \tag{2.27}$$

We shall often abuse notation and define $H^{[ij]}$ away from the overlap $\hat{\mathcal{U}}_i \cap \hat{\mathcal{U}}_j$ (in particular when the two patches do not intersect) using analytic continuation and the first equation in (2.27) to interpolate from $\hat{\mathcal{U}}_i$ to $\hat{\mathcal{U}}_j$. Ambiguities in the choice of path can be dealt with on a case by case basis.

The gluing conditions (2.26) are sufficient to determine $\tilde{\xi}^{[i]}_\Lambda, \alpha^{[i]}$ uniquely, up to overall real constants B_Λ, B_α which provide the extra $d+1$ coordinates mentioned above,

$$\begin{aligned}
\tilde{\xi}^{[i]}_\Lambda &= \frac{\mathrm{i}}{2} B_\Lambda + \frac{1}{2} \sum_j \oint_{C_j} \frac{\mathrm{d}\mathbf{z}'}{2\pi \mathrm{i} \mathbf{z}'} \frac{\mathbf{z}' + \mathbf{z}}{\mathbf{z}' - \mathbf{z}}\, \partial_{\xi^\Lambda} H^{[+j]}(\xi(\mathbf{z}')) + c_\Lambda \log \mathbf{z}\,, \qquad (2.28)\\
\alpha^{[i]} &= \frac{\mathrm{i}}{2} B_\alpha + \frac{1}{2} \sum_j \oint_{C_j} \frac{\mathrm{d}\mathbf{z}'}{2\pi \mathrm{i} \mathbf{z}'} \frac{\mathbf{z}' + \mathbf{z}}{\mathbf{z}' - \mathbf{z}} \left[H - \xi^\Lambda \partial_{\xi^\Lambda} H\right]^{[+j]} + c_\alpha \log \mathbf{z} + c_\Lambda \left(Y^\Lambda \mathbf{z}^{-1} + \bar{Y}^\Lambda \mathbf{z}\right)\,.
\end{aligned}$$

Here $\mathbf{z} \in \mathcal{U}_i$ and C_j is a contour surrounding $\mathcal{U}_j$, with $\mathcal{U}_i$ denoting the projection of $\hat{\mathcal{U}}_i$ to $\mathbb{C}P^1$. Eqs. (2.24) and (2.28) exhibit the complex coordinates on $\mathcal{Z}$ as functions of the coordinates $(\mathcal{R}, Y^a, \bar{Y}^a, A^\Lambda, B_I)$ on $\mathcal{M}$ and of the complex coordinate $\mathbf{z}$ on $\mathbb{C}P^1$, and parameterize the "contact twistor lines". Furthermore, in the toric case the potential $\Phi_{[i]}(x^\mu, \mathbf{z}) \equiv \Phi(x^\mu)$ is independent of $\mathbf{z}$ and the same in all patches,

$$e^\Phi = \frac{1}{4} \sum_j \oint_{C_j} \frac{\mathrm{d}\mathbf{z}'}{2\pi \mathrm{i} \mathbf{z}'} \left(\mathbf{z}'^{-1} Y^\Lambda - \mathbf{z}' \bar{Y}^\Lambda\right) \partial_{\xi^\Lambda} H^{[+j]}(\xi(\mathbf{z}')) + \frac{1}{2}\left(c_\Lambda A^\Lambda + c_\alpha\right)\,. \tag{2.29}$$

Note that due to consistency conditions (2.27), the index $[+]$ in (2.28), (2.29) can be replaced by any other patch index without affecting the result.

Let us mention also that for the purpose of expressing the metric on $\mathcal{M}$, it is sometimes more convenient to trade the coordinate $\mathcal{R}$ for the variable Φ. As we shall see below, this is natural for the hypermultiplet moduli space, since the contact potential Φ is identified with the four-dimensional dilaton ϕ.

2.3 Linear deformations

Deformations of a QK manifold $\mathcal{M}$ which preserve the QK property are controlled by the sheaf cohomology group $H^1(\mathcal{Z}, \mathcal{O}(2))$ [45, 46]. In practice, this means that they correspond to infinitesimal perturbations of the complex contact structure obtained by replacing

$$H^{[ij]}(\xi_{[i]}) \to H^{[ij]}(\xi^\Lambda_{[i]}) + H^{[ij]}_{(1)}(\xi^\Lambda_{[i]}, \tilde{\xi}^{[j]}_\Lambda, \alpha^{[j]}) \tag{2.30}$$

in (2.25), preserving the co-cycle conditions, reality conditions and modulo local contact transformations as in (2.27) (where now all quantities are functions of $(\xi^\Lambda_{[i]}, \tilde{\xi}^{[j]}_\Lambda, \alpha^{[j]})$) [21]. The function $H^{[ij]}_{(1)}$, holomorphic on $\hat{\mathcal{U}}_i \cap \hat{\mathcal{U}}_j$, corresponds to the contact Hamiltonian (or moment map) of the infinitesimal contact transformation performed in gluing the patches

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$\hat{\mathcal{U}}_i$ and $\hat{\mathcal{U}}_j$. As mentioned below (2.27), we abuse notation and consider $H_{(1)}^{[ij]}$ even when the patches do not intersect.

In general (for $\tilde{\xi}_\Lambda^{[j]}$ and $\alpha^{[j]}$-dependent $H_{(1)}^{[ij]}$) the perturbations break the $d+1$ isometries, and the position coordinate $\xi_{[i]}^\Lambda$ is no longer a global $\mathcal{O}(2)$ section. Indeed, the contact transformations (2.11) become, to linear order in the perturbation,

$$
\begin{aligned}
\xi_{[i]}^\Lambda &= \xi_{[j]}^\Lambda - T_{[ij]}^\Lambda , & \tilde{\xi}_\Lambda^{[i]} &= \tilde{\xi}_\Lambda^{[j]} - \tilde{T}_\Lambda^{[ij]} , \\
\alpha^{[i]} &= \alpha^{[j]} - \tilde{T}_\alpha^{[ij]} , & \Phi_{[i]} &= \Phi_{[j]} - \partial_{\alpha^{[j]}} H_{(1)}^{[ij]} ,
\end{aligned}
\tag{2.31}
$$

where we denoted

$$
\begin{aligned}
T_{[ij]}^\Lambda &\equiv - \partial_{\tilde{\xi}_\Lambda^{[j]}} H_{(1)}^{[ij]} + \xi_{[i]}^\Lambda \, \partial_{\alpha^{[j]}} H_{(1)}^{[ij]} , \qquad \tilde{T}_\Lambda^{[ij]} \equiv \partial_{\xi_{[i]}^\Lambda} \left(H^{[ij]} + H_{(1)}^{[ij]} \right) , \\
\tilde{T}_\alpha^{[ij]} &\equiv \left(H^{[ij]} + H_{(1)}^{[ij]} \right) - \xi_{[i]}^\Lambda \partial_{\xi_{[i]}^\Lambda} \left(H^{[ij]} + H_{(1)}^{[ij]} \right) .
\end{aligned}
\tag{2.32}
$$

In eq. (2.45) below, we interpret $(T_{[ij]}^\Lambda, \tilde{T}_\Lambda^{[ij]}, \tilde{T}_\alpha^{[ij]})$ as the contact vector field derived from the contact Hamiltonian $H^{[ij]} + H_{(1)}^{[ij]}$.

In the linear approximation the arguments of $H_{(1)}^{[ij]}$ can be taken to be the unperturbed contact twistor lines defined in (2.24) and (2.28). It is then straightforward to compute the correction to these unperturbed quantities,

$$
\begin{aligned}
\xi_{[i]}^\Lambda(\mathbf{z}, x^\mu) &= A^\Lambda + \mathbf{z}^{-1} Y^\Lambda - \mathbf{z} \bar{Y}^\Lambda + \frac{1}{2} \sum_j \oint_{C_j} \frac{\mathrm{d}\mathbf{z}'}{2\pi \mathrm{i} \mathbf{z}'} \, \frac{\mathbf{z}' + \mathbf{z}}{\mathbf{z}' - \mathbf{z}} \, T_{[+j]}^\Lambda(\mathbf{z}') , \\
\tilde{\xi}_\Lambda^{[i]}(\mathbf{z}, x^\mu) &= \frac{\mathrm{i}}{2} B_\Lambda + \frac{1}{2} \sum_j \oint_{C_j} \frac{\mathrm{d}\mathbf{z}'}{2\pi \mathrm{i} \mathbf{z}'} \, \frac{\mathbf{z}' + \mathbf{z}}{\mathbf{z}' - \mathbf{z}} \, \tilde{T}_\Lambda^{[+j]}(\mathbf{z}') + c_\Lambda \log \mathbf{z} , \\
\alpha^{[i]}(\mathbf{z}, x^\mu) &= \frac{\mathrm{i}}{2} B_\alpha + \frac{1}{2} \sum_j \oint_{C_j} \frac{\mathrm{d}\mathbf{z}'}{2\pi \mathrm{i} \mathbf{z}'} \, \frac{\mathbf{z}' + \mathbf{z}}{\mathbf{z}' - \mathbf{z}} \, \tilde{T}_\alpha^{[+j]}(\mathbf{z}') + c_\alpha \log \mathbf{z} + c_\Lambda \left(Y^\Lambda \mathbf{z}^{-1} + \bar{Y}^\Lambda \mathbf{z} \right) ,
\end{aligned}
\tag{2.33}
$$

where $\mathbf{z}$ is assumed to lie inside the contour C_i. Finally, the contact potential is now given by

$$
\begin{aligned}
e^{\Phi_{[i]}} = &\left(\frac{1}{4} \sum_j \oint_{C_j} \frac{\mathrm{d}\mathbf{z}'}{2\pi \mathrm{i} \mathbf{z}'} \left(\mathbf{z}'^{-1} Y^\Lambda - \mathbf{z}' \bar{Y}^\Lambda \right) \tilde{T}_\Lambda^{[+j]}(\xi(\mathbf{z}'), \tilde{\xi}(\mathbf{z}')) + \frac{1}{2} \left(c_\Lambda A^\Lambda + c_\alpha \right) \right) \\
&\times \left(1 + \frac{1}{2} \sum_j \oint_{C_j} \frac{\mathrm{d}\mathbf{z}'}{2\pi \mathrm{i} \mathbf{z}'} \, \frac{\mathbf{z}' + \mathbf{z}}{\mathbf{z}' - \mathbf{z}} \, \partial_{\alpha^{[j]}} H_{(1)}^{[+j]}(\mathbf{z}') \right) .
\end{aligned}
\tag{2.34}
$$

The equations (2.33), (2.34) provide sufficient information to compute the deformed QK metric on $\mathcal{M}$, using the procedure outlined at the end of Subsection 2.1. As mentioned below (2.29), it may be convenient to trade the coordinate $\mathcal{R}$ for the variable ϕ, defined in the perturbed case by

$$
\phi \equiv \mathrm{Re} \left[\Phi_{[+]}(\mathbf{z} = 0) \right] . \tag{2.35}
$$

Note that in general, the contact potentials $\Phi_{[i]}$ are functions of x^μ and $\mathbf{z}$. When $H_{(1)}^{[+j]}$ is independent of $\alpha^{[j]}$, however, the analysis simplifies considerably. As noted below (2.16), in this case the contact potentials $\Phi_{[i]}$ become all equal to a single real function on $\mathcal{M}$, which is nothing else but the function ϕ defined in (2.35). As we shall see, this situation prevails for D-instanton corrections to the hypermultiplet branch, but instanton corrections with non-vanishing NS5-brane charge necessitate the general formalism given here.

We end this executive summary of [21] with the following comment. All integration contours C_j appearing in the formulae above are closed since they surround open patches. It is however possible to generalize (2.33), (2.34) to open contours, provided the corresponding transition functions $H^{[+j]}$ are finite at the endpoints. This situation typically arises if one starts with a transition function with a branch cut in the patch $\mathcal{U}_j$, and shrinks the contour around $\mathcal{U}_j$ such that it surrounds the cut: the contribution reduces to the integral of the discontinuity of the transition function (or appropriate combinations thereof) along the cut. In this case the results (2.33), (2.34) acquire additional boundary contributions due to partial integrations, unless $H^{[+j]}$ vanishes at the endpoints of the open contour C_j. For example, if C_j is an open contour from $\mathbf{z}=0$ to $\mathbf{z}=\infty$, the following term must be added to (2.34),

$$\frac{1}{8\pi \mathrm{i}}\left(H^{[+j]}|_{\mathbf{z}=0}+H^{[+j]}|_{\mathbf{z}=\infty}\right). \tag{2.36}$$

Thus, instead of assigning a set of open patches and transition functions, a QK manifold can be characterized by providing a set of (closed or open) contours on $\mathbb{C}P^1$ and a set of associated functions $H^{[+j]}$. We will encounter such a description in the discussion of the instanton corrected hypermultiplet moduli space in section 4.

2.4 Action of continuous isometries

We now discuss how isometries on $\mathcal{M}$ lift to holomorphic isometries on its twistor space $\mathcal{Z}$. This issue has been discussed in the literature before, see e.g. [44, 47, 48]. Here we adapt it to our framework and make some additional observations.

Suppose $\mathcal{M}$ admits a continuous isometry generated by a Killing vector field κ. Generically, such an isometry rotates the quaternionic two-forms $\vec{\omega}$ (2.22) among each other,

$$\mathcal{L}_\kappa\vec{\omega}+\vec{r}\times\vec{\omega}=0\,, \tag{2.37}$$

where $\mathcal{L}$ denotes the Lie derivative and $\vec{r}$ generates the rotation of the quaternionic two-forms. The requirement that the two-forms are covariantly closed, $\mathrm{d}\vec{\omega}+\vec{p}\times\vec{\omega}=0$, determines the action of the Killing vector on the SU(2) connection

$$\mathcal{L}_\kappa\vec{p}=\mathrm{d}\vec{r}+\vec{r}\times\vec{p}\,. \tag{2.38}$$

Under the action of the isometry κ, the second term in the twistor space metric (2.1) is invariant, but the projectivized connection $\mathcal{P}\equiv D\mathbf{z}-\mathrm{d}\mathbf{z}$ transforms due to (2.38). This can be remedied by combining the Killing action κ on $\mathcal{M}$ with a compensating SU(2) rotation on the $\mathbb{C}P^1$ fiber, into the vector field $\kappa_{\mathcal{Z}}$ on $\mathcal{Z}$

$$\kappa_{\mathcal{Z}}=\kappa+\left(r_+-\mathrm{i}\,r_3\,\mathbf{z}+r_-\mathbf{z}^2\right)\partial_{\mathbf{z}}+\left(r_-+\mathrm{i}\,r_3\,\bar{\mathbf{z}}+r_+\bar{\mathbf{z}}^2\right)\partial_{\bar{\mathbf{z}}}\,. \tag{2.39}$$

Under the Lie action of $\kappa_{\mathcal{Z}}$, the canonical one-form $D\mathbf{z}$ transforms as

$$\mathcal{L}_{\kappa_{\mathcal{Z}}}D\mathbf{z}=\left(-\mathrm{i}\,r_3+2\,r_-\mathbf{z}\right)D\mathbf{z}\,, \tag{2.40}$$

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which ensures the invariance of the metric (2.1). The vector $\kappa_{\mathcal{Z}}$ is in fact the real part of a holomorphic vector field κ_h on $\mathcal{Z}$, in accord with the fact that any isometric action on $\mathcal{M}$ can be lifted to an holomorphic action on $\mathcal{Z}$ [44, 47, 48].

The vector $\vec{r}$ is related to the vector-valued moment map $\vec{\mu}$ via [44]

$$\vec{\mu} = \frac{1}{2}(\vec{r} + \kappa \cdot \vec{p}) \,, \tag{2.41}$$

where $\vec{p}$ is the SU(2) connection and the dot denotes the inner product. The moment map provides a global holomorphic section of $H^0(\mathcal{Z}, \mathcal{O}(2))$,

$$\mu_{[i]} \equiv e^{\Phi_{[i]}} \left(\mu_+ \, \mathbf{z}^{-1} - \mathrm{i}\mu_3 + \mu_- \mathbf{z} \right) \,. \tag{2.42}$$

To see that $\mu_{[i]}$ is holomorphic, note that by virtue of (2.39), (2.40) and (2.41), it equals the inner product of the Killing vector $\kappa_{\mathcal{Z}}$ with the holomorphic one-form $\mathcal{X}^{[i]}$,

$$\kappa_{\mathcal{Z}} \cdot \mathcal{X}^{[i]} = \mu_{[i]} \,. \tag{2.43}$$

Since $\kappa_{\mathcal{Z}}$ is the real part of a holomorphic vector field, $\mu_{[i]}$ is indeed a holomorphic function on $\hat{\mathcal{U}}_i$, hence defines an element of $H^0(\mathcal{Z}, \mathcal{O}(2))$. Conversely, it is known that any element of $H^0(\mathcal{Z}, \mathcal{O}(2))$ determines a continuous isometry of $\mathcal{M}$ [14].

In fact, (2.43) identifies $\mu_{[i]}$ as the contact Hamiltonian for the contact vector field $\kappa_{\mathcal{Z}}$ [43], and $\mu^{\mathcal{S}}_{[i]} \equiv \nu^{\alpha}_{[i]} \mu_{[i]}$ as (minus) the complex moment map for the lift $\kappa_{\mathcal{S}}$ of κ to the Swann bundle. The Poisson bracket associated to the complex symplectic structure on $\mathcal{Z}_{\mathcal{S}}$ descends to a "contact Poisson" bracket on $\mathcal{Z}$, mapping two local sections (μ_1, μ_2) of $\mathcal{O}(2m) \times \mathcal{O}(2n)$ to a local section of $\mathcal{O}(2(m+n-1))$,

$$\begin{aligned} \{\mu_1, \mu_2\} \equiv & \, m \, \mu_1 \partial_\alpha \mu_2 + \partial_\alpha \mu_1 \, \xi^\Lambda \partial_{\xi^\Lambda} \mu_2 + \partial_{\xi^\Lambda} \mu_1 \partial_{\tilde{\xi}_\Lambda} \mu_2 \\ & - n \, \mu_1 \partial_\alpha \mu_2 - \partial_\alpha \mu_2 \, \xi^\Lambda \partial_{\xi^\Lambda} \mu_1 - \partial_{\xi^\Lambda} \mu_2 \partial_{\tilde{\xi}_\Lambda} \mu_1 \,. \end{aligned} \tag{2.44}$$

For $m = n = 1$, this defines a standard Poisson bracket on $H^0(\mathcal{Z}, \mathcal{O}(2))$, such that, for two contact vector fields $\kappa_{1,2}$, $\mu_{[\kappa_1, \kappa_2]} = \{\mu_{\kappa_1}, \mu_{\kappa_2}\}$. For $m = 1, n = 0$, one obtains the action of the Killing vector $\kappa_{\mathcal{Z}}$ on the local complex coordinates,

$$\{\mu, \xi^\Lambda\} = -\partial_{\tilde{\xi}_\Lambda} \mu + \xi^\Lambda \partial_\alpha \mu \,, \qquad \{\mu, \tilde{\xi}_\Lambda\} = \partial_{\xi^\Lambda} \mu \,. \qquad \{\mu, \alpha\} = \mu - \xi^\Lambda \partial_{\xi^\Lambda} \mu \,. \tag{2.45}$$

This reproduces the vector field $(T^\Lambda_{[ij]}, \tilde{T}^{[ij]}_\Lambda, \tilde{T}^{[ij]}_\alpha)$ in (2.32) for $\mu = H^{[ij]} + H^{[ij]}_{(1)}$.

Finally, inserting (2.40) into (2.3), the holomorphic one-form transforms as

$$\mathcal{L}_{\kappa_{\mathcal{Z}}} \mathcal{X}^{[i]} = \left(\kappa_{\mathcal{Z}} \cdot \Phi_{[i]} + r_- \mathbf{z} - r_+ \mathbf{z}^{-1} \right) \mathcal{X}^{[i]} = \left(\partial_{\alpha^{[i]}} \mu_{[i]} \right) \mathcal{X}^{[i]} \,. \tag{2.46}$$

This determines the variation of the contact and Kähler potentials to be

$$\kappa_{\mathcal{Z}} \cdot \Phi_{[i]} = \partial_{\alpha^{[i]}} \mu_{[i]} - r_- \mathbf{z} + r_+ \mathbf{z}^{-1} \,, \qquad \kappa_{\mathcal{Z}} \cdot K^{[i]}_{\mathcal{Z}} = \mathrm{Re} \left(\partial_{\alpha^{[i]}} \mu_{[i]} \right) \,. \tag{2.47}$$

2.5 Relation to Swann's construction

As mentioned in the introduction, QK manifolds $\mathcal{M}$ are in one-to-one correspondence with hyperkähler cones $\mathcal{S}$ via the superconformal quotient and Swann's constructions [15, 18]. Such cones are completely characterized by a single function, the hyperkähler potential χ, which is a Kähler potential for the whole sphere of complex structures on $\mathcal{S}$. In [21] it was shown that χ is related to the contact potential via

$$\chi = \frac{e^{\phi}}{4r^{\flat}}\,, \tag{2.48}$$

where ϕ is defined in (2.35) and $r^{\flat}$ is a certain function invariant under the SU(2) isometric action on $\mathcal{S}$, with weight one under dilations (see [21] for more details).

The advantage of χ over ϕ is that it is invariant under all isometries of $\mathcal{M}$, while ϕ transforms non-trivially according to (2.47). This invariance was instrumental in the previous studies of instanton corrections [32, 34], which are easily translated into our framework via (2.48).

3 Perturbative hypermultiplet moduli spaces

In this section we recall some known results on the perturbative hypermultiplet moduli space in Type IIA and IIB string theories compactified on a CY threefold, and phrase them in the language of twistors and complex contact geometry.

3.1 Type IIA compactified on a CY threefold X

The hypermultiplet moduli space $\mathcal{M}_{\rm HM}^{A}$ in Type IIA string theory compactified on a CY threefold X is a QK manifold of real dimension $d = 4(h_{2,1}(X)+1)$ [49–51]. It describes the dynamics of the complex structure moduli $X^\Lambda = \int_{\gamma^\Lambda}\Omega$, $F_\Lambda = \int_{\gamma_\Lambda}\Omega$, the RR scalars

$$\zeta^\Lambda = \int_{\gamma^\Lambda} A^{(3)}\,, \qquad \tilde\zeta_\Lambda = \int_{\gamma_\Lambda} A^{(3)}\,, \tag{3.1}$$

the four-dimensional dilaton $e^\phi = 1/g_{(4)}^2$ and the Neveu-Schwarz (NS) axion σ, dual to the Neveu-Schwarz two-form B in four dimensions. Here γ^Λ and γ_Λ form a symplectic basis of A and B cycles in $H_3(X,\mathbb{Z})$, with intersection product $\langle\gamma^\Lambda,\gamma_\Sigma\rangle = \delta^\Lambda_\Sigma$.

To all orders in perturbation theory, the metric on $\mathcal{M}_{\rm HM}^{A}$ admits a $2d+1$-dimensional Heisenberg group of tri-holomorphic isometries, corresponding to translations along the RR potentials $(\zeta^\Lambda,\tilde\zeta_\Lambda)$ and the NS axion σ. Thus, it falls into the class of toric QK geometries discussed in section 2.2. These continuous isometries are in general broken to a discrete subgroup by instanton corrections. At tree level, the geometry of $\mathcal{M}_{\rm HM}^{A}$ is obtained from the moduli space $\mathcal{M}_{\rm cs}$ of complex structure deformations of X (which would correspond to the vector multiplet moduli space of Type IIB string theory compactified on the *same* CY X) via the "c-map" construction [49, 50]. $\mathcal{M}_{\rm cs}$ is completely characterized by the prepotential $F(X^\Lambda)$, a homogeneous function of degree 2 of the A-type periods X^Λ, such that the B-type periods are given by $F_\Lambda = \partial F/\partial X^\Lambda$. X^Λ provide a set of homogeneous coordinates on $\mathcal{M}_{\rm cs}$, and (away from the vanishing locus of X^0) may be traded for the inhomogeneous coordinates $z^a = X^a/X^0$. At one-loop, the c-map metric on $\mathcal{M}$ receives a correction proportional to the Euler class $\chi_X = 2(h^{1,1}(X) - h^{2,1}(X))$.

The twistor space $\mathcal{Z}$ of the QK manifold $\mathcal{M}_{\rm HM}^A$ admits the following simple description. $\mathcal{Z}$ can be covered by two patches $\hat{\mathcal{U}}_+, \hat{\mathcal{U}}_-$ which project to open disks centered around $\mathbf{z}=0$ and $\mathbf{z}=\infty$ on $\mathbb{C}P^1$, and a third patch $\hat{\mathcal{U}}_0$ which projects to the rest of $\mathbb{C}P^1$. The transition functions between complex Darboux coordinates on each patch are given by [21]

$$H^{[0+]} = -\frac{\mathrm{i}}{2}\, F(\xi^\Lambda)\,, \qquad H^{[0-]} = -\frac{\mathrm{i}}{2}\, \bar F(\xi^\Lambda)\,, \qquad c_\alpha = \frac{\chi_X}{96\pi}\,, \tag{3.2}$$

with the other anomalous dimensions $c_\Lambda = 0$. The non-vanishing anomalous dimension c_α incorporates the effect of the one-loop correction. Based on the string theory amplitudes [29, 30], the QK metric obtained from (3.2) was calculated in [27, 28]. It is believed to be the correct metric on $\mathcal{M}$ to all orders in perturbation theory [21, 27, 30]. At this perturbative level, the coordinates Y^a, A^Λ, B_I introduced on general grounds in section 2 are related to the Type IIA variables via

$$\zeta^\Lambda = A^\Lambda\,, \quad \tilde\zeta_\Lambda = B_\Lambda + A^\Sigma\, \mathrm{Re}\, F_{\Lambda\Sigma}(z)\,, \quad \sigma = -2B_\alpha - A^\Lambda B_\Lambda\,, \quad Y^a = \mathcal{R}\, z^a\,, \tag{3.3}$$

where $\mathcal{R}$ may be expressed in terms of the contact potential by means of (2.29),

$$e^{\Phi_{\rm pert}} = \frac{\mathcal{R}^2}{4}\, K(z,\bar z) + \frac{\chi_X}{192\pi} \tag{3.4}$$

with $K(z,\bar z) \equiv -2\,\mathrm{Im}\,(\bar z^\Lambda F_\Lambda)$. The contact potential $\Phi_{\rm pert}$ is in turn identified with the 4D dilaton ϕ. Denoting

$$\rho_\Lambda \equiv -2\mathrm{i}\tilde\xi_\Lambda^{[0]}\,, \qquad \tilde\alpha \equiv 4\mathrm{i}\alpha^{[0]} + 2\mathrm{i}\tilde\xi_\Lambda^{[0]}\xi^\Lambda\,, \tag{3.5}$$

the contact twistor lines in the patch $\hat{\mathcal{U}}_0$ are given by [21, 52]

$$\begin{aligned} \xi^\Lambda &= \zeta^\Lambda + \mathcal{R}\left(\mathbf{z}^{-1} z^\Lambda - \mathbf{z}\,\bar z^\Lambda\right)\,, \\ \rho_\Lambda &= \tilde\zeta_\Lambda + \mathcal{R}\left(\mathbf{z}^{-1} F_\Lambda(z) - \mathbf{z}\,\bar F_\Lambda(\bar z)\right)\,, \\ \tilde\alpha &= \sigma + \mathcal{R}\left(\mathbf{z}^{-1} W(z) - \mathbf{z}\,\bar W(\bar z)\right) + \frac{\mathrm{i}\chi_X}{24\pi}\,\log \mathbf{z}\,, \end{aligned} \tag{3.6}$$

where

$$W(z) \equiv F_\Lambda(z)\zeta^\Lambda - z^\Lambda\tilde\zeta_\Lambda\,. \tag{3.7}$$

Electric-magnetic duality acts on $\mathcal{Z}$ by complex contact transformations

$$\begin{pmatrix}\xi^\Lambda \\ \rho_\Lambda\end{pmatrix} \mapsto \begin{pmatrix}\mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D}\end{pmatrix}\begin{pmatrix}\xi^\Lambda \\ \rho_\Lambda\end{pmatrix}\,, \qquad \tilde\alpha \mapsto \tilde\alpha\,, \tag{3.8}$$

where $\left(\begin{smallmatrix}\mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D}\end{smallmatrix}\right)$ is a $\mathrm{Sp}(2h_{1,2}(X),\mathbb{Z})$ matrix whose block matrices satisfy

$$\mathcal{A}^{\rm T}\mathcal{C} - \mathcal{C}^{\rm T}\mathcal{A} = \mathcal{B}^{\rm T}\mathcal{D} - \mathcal{D}^{\rm T}\mathcal{B} = 0\,, \qquad \mathcal{A}^{\rm T}\mathcal{D} - \mathcal{C}^{\rm T}\mathcal{B} = \mathbf{1}\,. \tag{3.9}$$

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This action is in general not an isometry of $\mathcal{M}$, since the moment map associated to an infinitesimal action (3.8) with $\mathcal{B} = \mathcal{B}^{\mathrm{T}}$, $\mathcal{C} = \mathcal{C}^{\mathrm{T}}$, $\mathcal{A}^{\mathrm{T}} + \mathcal{D} = 0$, given by

$$\mu = -\tilde{\xi}_{\Lambda}^{[0]} \mathcal{A}^{\Lambda}_{\Sigma} \xi^{\Sigma} + \mathrm{i}\, \tilde{\xi}_{\Lambda}^{[0]} \mathcal{B}^{\Lambda\Sigma} \tilde{\xi}_{\Sigma}^{[0]} + \frac{\mathrm{i}}{4} \xi^{\Lambda} \mathcal{C}_{\Lambda\Sigma} \xi^{\Sigma} , \tag{3.10}$$

is in general *not* a global $\mathcal{O}(2)$ section.[7]

3.2 Type IIB compactified on a CY threefold Y

The hypermultiplet moduli space $\mathcal{M} = \mathcal{M}_{\mathrm{HM}}^{B}$ in Type IIB string theory compactified on a CY threefold Y is a QK manifold of real dimension $d = 4(h^{1,1}(Y) + 1)$ [30, 49, 50, 54]. It describes the dynamics of the Kähler moduli $z^a \equiv b^a + \mathrm{i} t^a = \int_{\gamma^a} \mathcal{J}$, the RR scalars[8]

$$\begin{aligned} c^0 &= A^{(0)} , \qquad c^a = \int_{\gamma^a} A^{(2)} , \qquad c_a = -\int_{\gamma_a} \left(A^{(4)} - \frac{1}{2} B \wedge A^{(2)} \right) , \\ c_0 &= -\int_Y \left(A^{(6)} - B \wedge A^{(4)} + \frac{1}{3} B \wedge B \wedge A^{(2)} \right) , \end{aligned} \tag{3.11}$$

the four-dimensional dilaton ϕ and the NS axion ψ, dual to the NS 2-form B in four dimensions. Here $\mathcal{J} \equiv B + \mathrm{i}\, J = z^a \omega_a$ is the complexified Kähler form on Y. Furthermore, γ^a, $a = 1, \dots, h^{1,1}(Y)$, denote a basis of 2-cycles (Poincaré dual to 4-forms ω^a), and γ_a a basis of 4-cycles (Poincaré dual to 2-forms ω_a), such that

$$\omega_a \wedge \omega_b = \kappa_{abc} \omega^c , \qquad \omega_a \wedge \omega^b = \delta_a^b \omega_Y , \qquad \int_{\gamma^a} \omega_b = \int_{\gamma_b} \omega^a = \delta_b^a , \tag{3.12}$$

where ω_Y is the volume form, normalized to $\int_Y \omega_Y = 1$, and $\kappa_{abc} = \int_Y \omega_a \omega_b \omega_c = \langle \gamma_a, \gamma_b, \gamma_c \rangle$ is the triple intersection product in $H_4(Y, \mathbb{Z})$. In the large volume limit, the 4D dilaton ϕ is related to the 10D string coupling g_s via $e^{\phi} = V(t^a)/g_s^2$, where $V(t^a) = \frac{1}{6} \int_Y J \wedge J \wedge J = \frac{1}{6} \kappa_{abc} t^a t^b t^c$ is the volume of Y in string units. The ten-dimensional coupling $\tau_2 \equiv 1/g_s$ and the RR axion $\tau_1 \equiv c^0$ can be combined into the ten-dimensional axio-dilaton field $\tau = \tau_1 + \mathrm{i} \tau_2$.

As in Type IIA string theory, to all orders in perturbation theory, the metric on $\mathcal{M}_{\mathrm{HM}}^{B}$ admits a $2d+1$-dimensional Heisenberg group of isometries, corresponding to translations along the RR potentials $(c^{\Lambda}, c_{\Lambda})$ and the NS axion ψ. At tree level, it is obtained from the moduli space $\mathcal{M}_{\mathrm{ks}}$ of complexified Kähler deformations (which would correspond to the vector multiplet moduli space in Type IIA string theory compactified on the *same* CY Y) via the c-map. $\mathcal{M}_{\mathrm{ks}}$ is again characterized by the prepotential $F(X^{\Lambda})$, which now receives world-sheet instanton corrections. The prepotential has the standard large volume

[7] For special choices of κ_{abc}, related to Jordan algebras of degree 3, a subgroup of $\mathrm{Sp}(2h_{1,2}(X), \mathbb{Z})$ may however act isometrically, see e.g. the $\mathrm{SL}(2, \mathbb{R})$ generators Y_+, Y_0, Y_- in eq. (3.50) of [53] for the special case $\mathcal{M} = G_{2(2)}/\mathrm{SO}(4)$.

[8] The B-dependent corrections ensure that these fields have simple transformation properties under S-duality, see eq. (3.16) below. The correction to c_a appears in [55], footnote 14, the correction to c_0 seems to be novel.

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expansion (in the conventions of [33], up to a sign change in κ_{abc})),

$$F(X^\Lambda) = -\kappa_{abc}\,\frac{X^a X^b X^c}{6X^0} + \chi_Y\,\frac{\zeta(3)(X^0)^2}{2(2\pi \mathrm{i})^3} - \frac{(X^0)^2}{(2\pi \mathrm{i})^3} \sum_{k_a\gamma^a \in H_2^+(Y)} n^{(0)}_{k_a}\,\mathrm{Li}_3\left(e^{2\pi \mathrm{i} k_a X^a/X^0}\right), \tag{3.13}$$

where k_a runs over effective homology classes (i.e. $k_a \geq 0$ for all a, not all of them vanishing simultaneously), $n^{(0)}_{k_a}$ is the genus zero BPS invariant in the homology class $k_a\gamma^a \in H_2^+(Y,\mathbb{Z})$, $\mathrm{Li}_s(x) = \sum_{m=1}^\infty m^{-s}x^m$ is the polylogarithm function, and χ_Y is the Euler number of Y. Note that the last two terms in (3.13) may be combined by including the zero class $k_a = 0$ in the sum and setting $n^{(0)}_0 = -\chi_Y/2$.

At one-loop, the c-map metric on $\mathcal{M}$ receives a correction proportional to the Euler class χ_Y of Y. The twistor space $\mathcal{Z}$ is described by the same transition functions (3.2) and contact potential in (3.4), with χ_X replaced by $-\chi_Y$. Using the large volume expansion (3.13) and identifying $\mathcal{R} = \tau_2/2$ (as will become clear in (3.20) below), the contact potential can be further expressed as

$$e^{\Phi_{\rm pert}} = \frac{\tau_2^2}{2}\,V(t^a) - \frac{\chi_Y\zeta(3)}{8(2\pi)^3}\,\tau_2^2 + e^{\Phi_{\rm ws}} - \frac{\chi_Y}{192\pi}\,, \tag{3.14}$$

where

$$e^{\Phi_{\rm ws}} = \frac{\tau_2^2}{4(2\pi)^3} \sum_{k_a\gamma^a \in H_2^+(Y)} n^{(0)}_{k_a}\,\mathrm{Re}\left[\mathrm{Li}_3\left(e^{2\pi \mathrm{i} k_a z^a}\right) + 2\pi k_a t^a\,\mathrm{Li}_2\left(e^{2\pi \mathrm{i} k_a z^a}\right)\right] \tag{3.15}$$

is the world-sheet instanton contribution. In the large volume limit, $\Phi_{\rm pert}$ coincides with the 4D dilaton ϕ, and may in fact be taken as the definition of the 4D dilaton in the quantum regime.

When X and Y are related by mirror symmetry (which requires $\chi_X = -\chi_Y$), the hypermultiplet moduli spaces $\mathcal{M}^A_{\rm HM}$ and $\mathcal{M}^B_{\rm HM}$ must be identical, with the Kähler moduli $z^a = b^a + \mathrm{i}t^a$ of Y being identified with the complex structure moduli of Y. The relation between the Type IIA variables $(\mathcal{R}, Y^a, \zeta^\Lambda, \tilde\zeta_\Lambda, \sigma)$ and the Type IIB variables $(\tau, z^a, c^a, c_a, c_0, \psi)$ will be obtained in the next subsection from S-duality.

3.3 S-duality and mirror map

At the classical level, i.e., at tree-level and leading order in the α' expansion, Type IIB supergravity in ten dimensions is invariant under a continuous $\mathrm{SL}(2,\mathbb{R})$ symmetry. After compactification on Y and in the large volume limit, the metric on the hypermultiplet moduli space $\mathcal{M} \equiv \mathcal{M}^B_{\rm HM}$ admits an isometry group $\mathrm{SL}(2,\mathbb{R})$, acting as

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}\,, \qquad t^a \mapsto t^a|c\tau + d|\,, \qquad c_a \mapsto c_a\,, \tag{3.16}$$
$$\begin{pmatrix} c^a \\ b^a \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} c^a \\ b^a \end{pmatrix}, \qquad \begin{pmatrix} c_0 \\ \psi \end{pmatrix} \mapsto \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}\begin{pmatrix} c_0 \\ \psi \end{pmatrix}$$

with $ad - bc = 1$. While the existence of this isometric action was established in [30, 56], it is instructive to derive it again by twistorial methods.

For this purpose, it suffices, as explained below (2.42), to construct global $\mathcal{O}(2)$ sections whose Poisson brackets satisfy the $\mathrm{SL}(2,\mathbb{R})$ algebra. The following three quantities[9]

$$\mu^+ = -\tilde{\xi}_0^{[+]} - \frac{\mathrm{i}}{12(\xi^0)^2}\,\kappa_{abc}\,\xi^a\xi^b\xi^c\,, \quad \mu^0 = \alpha^{[+]} - \xi^0\tilde{\xi}_0^{[+]}\,, \qquad \mu^- = \alpha^{[+]}\xi^0 \tag{3.17}$$

satisfy these requirements. Indeed, they are manifestly regular at $\mathbf{z}=0$, (except for the $1/\mathbf{z}$ pole that we allow for global $\mathcal{O}(2)$ sections, as in e.g. (2.24)). The apparent singularity at the zeros of ξ^0 can be removed by rewriting them in the patch $\hat{\mathcal{U}}_0$, using the classical limit of the transition functions (3.2),

$$\mu^+ = -\tilde{\xi}_0^{[0]}\,, \qquad \mu^0 = \alpha^{[0]} - \xi^0\tilde{\xi}_0^{[0]}\,, \qquad \mu^- = \alpha^{[0]}\xi^0 - \frac{\mathrm{i}}{12}\kappa_{abc}\,\xi^a\xi^b\xi^c\,. \tag{3.18}$$

The regularity at $\mathbf{z}=\infty$ is of course guaranteed by the reality condition. Exponentiating the infinitesimal action generated by the contact Hamiltonians (3.18), we arrive at the $\mathrm{SL}(2,\mathbb{R})$ action on the contact twistor lines in the patch $\hat{\mathcal{U}}_0$,

$$\begin{gathered}\xi^0 \mapsto \frac{a\xi^0+b}{c\xi^0+d}\,, \qquad \xi^a \mapsto \frac{\xi^a}{c\xi^0+d}\,, \qquad \tilde{\xi}_a \mapsto \tilde{\xi}_a + \frac{\mathrm{i}\,c}{4(c\xi^0+d)}\kappa_{abc}\xi^b\xi^c\,, \\ \begin{pmatrix}\tilde{\xi}_0\\ \alpha\end{pmatrix} \mapsto \begin{pmatrix} d & -c\\ -b & a\end{pmatrix}\begin{pmatrix}\tilde{\xi}_0\\ \alpha\end{pmatrix} + \frac{\mathrm{i}}{12}\kappa_{abc}\xi^a\xi^b\xi^c\begin{pmatrix} c^2/(c\xi^0+d)\\ -[c^2(a\xi^0+b)+2c]/(c\xi^0+d)^2\end{pmatrix}.\end{gathered} \tag{3.19}$$

The action on ξ^Λ agrees with the standard linear action on the complex coordinates ν^I on the Swann bundle [26, 32] after projectivizing. Under the action (3.19), the complex contact one-form transforms by an overall holomorphic factor $\mathcal{X}^{[i]} \to \mathcal{X}^{[i]}/(c\xi^0+d)$, leaving the complex contact structure invariant.

The holomorphic contact action (3.19) on $\mathcal{Z}$ descends to an isometric action on $\mathcal{M}$, and a $\mathrm{SU}(2)$ rotation on the fiber. It may be checked that (3.19) agrees with the standard action (3.16), provided one identifies[10]

$$\begin{aligned}&\mathcal{R} = \frac{1}{2}\,\tau_2\,, \qquad Y^a = \frac{1}{2}\,\tau_2\,z^a\,, \qquad \zeta^0 = \tau_1\,, \qquad \zeta^a = -(c^a - \tau_1 b^a)\,,\\ &\tilde{\zeta}_a = c_a + \frac{1}{2}\,\kappa_{abc}\,b^b(c^c-\tau_1 b^c)\,, \qquad \tilde{\zeta}_0 = c_0 - \frac{1}{6}\,\kappa_{abc}\,b^a b^b(c^c-\tau_1 b^c)\,,\\ &\sigma = -2(\psi + \frac{1}{2}\tau_1 c_0) + c_a(c^a-\tau_1 b^a) - \frac{1}{6}\,\kappa_{abc}\,b^a c^b(c^c-\tau_1 b^c)\,.\end{aligned} \tag{3.20}$$

These relations, valid in the classical limit, provide the "generalized mirror map" between the Type IIA variables $(\mathcal{R}, Y^a, \zeta^\Lambda, \tilde{\zeta}_\Lambda, \sigma)$ and the Type IIB variables $(\tau, b^a, t^a, c^a, c_a, c_0, \psi)$. They agree with the identification found by dimensional reduction of the Type IIB supergravity Lagrangian on Y in [30, 56].[11]

[9]These formulae agree with the moment maps computed in [53] for the special case $\mathcal{M} = G_{2(2)}/\mathrm{SO}(4)$; compare (3.18) with the generators $E_{p^0}, H+2Y_0, F_{p^0}$ in eq. (3.50) of [53].

[10]The coefficient of the first term in $\tilde{\zeta}_0$ and σ cannot be determined from $\mathrm{SL}(2,\mathbb{R})$ invariance alone, as it can be changed by a field redefinition shifting $(\tilde{\xi}_0,\alpha)$ by a term proportional to the doublet (c_0,ψ). It is fixed however by requiring the consistency of the D-brane actions (4.18) and (4.21) under mirror symmetry.

[11] Note that this identification was derived independently on the vector multiplet side in the one-modulus case in [57], eq. (3.13)–(3.14). The identifications are $(V,\rho_2,\rho_1,\mu_1,\mu_2,\nu,\tilde{\mu}_1,\tilde{\mu}_2)_{[57]} = (\tau_2(t^a)^2, e^{\phi/2}/\tau_2^{3/2}, -\tau_1, \sqrt{3}c^a, \sqrt{3}b^a, c_a/\sqrt{3}, -\psi/\sqrt{2}, c_0/\sqrt{2})_{\mathrm{Here}}$.

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Expressing $\mathbf{z}$ in terms of ξ^0 and using the first equation in (3.19), we obtain the action of $\mathrm{SL}(2,\mathbb{R})$ on the $\mathbb{C}P^1$ fiber,

$$\mathbf{z} \mapsto \frac{c\tau_2 + \mathbf{z}(c\tau_1 + d) + \mathbf{z}|c\tau + d|}{(c\tau_1 + d) + |c\tau + d| - \mathbf{z}c\tau_2} \,. \tag{3.21}$$

Moreover, the contact potential transforms as

$$e^{\Phi} \mapsto e^{\Phi}/|c\tau + d| \,, \tag{3.22}$$

which ensures that the Kähler potential varies by a Kähler transformation,

$$K_{\mathcal{Z}} \mapsto K_{\mathcal{Z}} - \log(|c\xi^0 + d|) \,. \tag{3.23}$$

In the presence of worldsheet instantons or after including the one-loop correction, the continuous isometries associated to μ^- and μ^0 are broken since their purported moment maps are no longer regular at $\mathbf{z} = 0$. As shown in [32, 34] and reviewed in section 4.1 below, it is possible to restore the invariance under a discrete subgroup $\mathrm{SL}(2,\mathbb{Z})$ by incorporating D-instanton corrections.

4 D-instanton corrections in Type II compactifications

In this section, we determine the form of all D-instanton corrections to the hypermultiplet metric, as linear perturbations around the perturbative QK metric. We start by reviewing and extending the results obtained in [32, 34] for the contribution of D1-D(-1)-instantons and A-type D2-brane instantons in Calabi-Yau compactifications of Type IIB and Type IIA strings, respectively. We then generalize these results to all D-instantons, using electric-magnetic duality and mirror symmetry. In Subsection 4.4 we extend our considerations beyond linear order.

4.1 S-duality and A-type D-instanton corrections

While (3.14) is believed to be the full perturbative result, it cannot be exact, since it is not consistent with $\mathrm{SL}(2,\mathbb{Z})$ duality of ten-dimensional Type IIB string theory. Indeed, the subleading terms in (3.14) spoil the transformation rule (3.22), and the Kähler potential no longer transforms by a Kähler transformation.

As explained in [32], the invariance under the discrete subgroup $\mathrm{SL}(2,\mathbb{Z})$ can be restored by summing over images, using similar techniques as the ones used for R^4 couplings in toroidal compactifications [58–61]. The result is expressed in terms of a generalized Eisenstein series,

$$e^{\Phi_{\rm inv}} = \frac{\tau_2^2}{2}\, V(t^a) + \frac{\sqrt{\tau_2}}{8(2\pi)^3} \sum_{k_a\gamma^a \in H_2^+(Y)\cup\{0\}} n^{(0)}_{k_a} \sum_{m,n}{}' \frac{\tau_2^{3/2}}{|m\tau + n|^3} \left(1 + 2\pi|m\tau + n|k_a t^a\right) e^{-S_{m,n,k_a}} \,, \tag{4.1}$$

where

$$S_{m,n,k_a} = 2\pi k_a |m\tau + n|\, t^a - 2\pi \mathrm{i} k_a (mc^a + nb^a) \tag{4.2}$$

and the primed sum runs over pairs of integers $(m,n) \neq (0,0)$. For $k_a = 0$ the sum encodes the perturbative contributions together with the D(-1)-instanton corrections, while for $k_a\gamma^a \in H_2^+(Y)$, the exponent S_{m,n,k_a} is the classical action of a (p,q)-string (or rather (m,n)-string) wrapped on the 2-cycle $k_a\gamma^a$.

Although the representation (4.1) makes S-duality invariance manifest, it cannot be directly interpreted as an instanton sum. To expose the instantons, it is advisable to perform a Poisson resummation on the integer n [58]. Denoting the dual integer by k_0, one obtains [34]

$$\begin{aligned} e^{\Phi_{\rm inv}} =& \frac{\tau_2^2}{2}\, V\left(t^a\right) - \frac{\sqrt{\tau_2}}{8(2\pi)^3}\,\chi_Y \left[\zeta(3)\,\tau_2^{3/2} + \frac{\pi^2}{3}\,\tau_2^{-1/2}\right] + e^{\Phi_{\rm ws}} \\ &+ \frac{\tau_2}{8\pi^2} {\sum_{k_\Lambda}}' n_{k_a}^{(0)} \sum_{m=1}^{\infty} \frac{|k_\Lambda z^\Lambda|}{m} \cos\left(2\pi m\, k_\Lambda \zeta^\Lambda\right) K_1\left(2\pi m\, |k_\Lambda z^\Lambda| \tau_2\right)\,, \end{aligned} \tag{4.3}$$

where we denoted $k_\Lambda = (k_0, k_a)$. In this equation, the sum runs over $k_0 \in \mathbb{Z}$, $k_a\gamma^a \in H_2^+(Y,\mathbb{Z})$ excluding the value $(k_0,k_a)=0$ (as indicated by the prime), and $e^{\Phi_{\rm ws}}$ is the worldsheet instanton contribution (3.15). The term in square brackets in (4.3) combines two perturbative contributions: the first is perturbative in the α' expansion and corresponds to the second term in (3.14), whereas the second is the one-loop contribution corresponding to the last term in (3.14). In contrast, the second line in (4.3) has a non-perturbative origin: it describes the contributions of "bound states" of m Euclidean D1-strings wrapping rational curves (counted by the Gopakumar-Vafa invariant $n_{k_a}^{(0)}$) in the homology class $k_a\gamma^a$ and $m\,k_0$ D(-1)-instantons, with classical action

$$S_{\rm cl} = 2\pi m \tau_2\, |k_\Lambda z^\Lambda| + 2\pi \mathrm{i} m\, k_\Lambda \zeta^\Lambda\,. \tag{4.4}$$

In the subsector $k_a = 0$, the sum reduces to D(-1)-instanton contributions, analogous to the ones appearing in R^4 couplings in ten dimensions [58].

Using the mirror map (3.20), the same result (4.3) can be interpreted from the point of view of Type IIA string theory compactified on X [34]: the classical instanton action (4.4) corresponds to a bound state of m D2-branes wrapping the A-cycle $k_\Lambda\gamma^\Lambda$ in $H_3(X,\mathbb{Z})$. By mirror symmetry, the BPS invariant $n_{k_a}^{(0)}$ of Y should count the number of special Lagrangian 3-cycles homologous to $k_\Lambda\gamma^\Lambda$ in $H_3(X,\mathbb{Z})$; in particular, this number should be independent of k_0. Of course, on the Type IIA side the restriction to A-cycles is artificial, and will be relaxed in the next subsection.

Leaving a more detailed discussion of the instanton effects to section 4.3, we now discuss how the contact potential (4.3) may be understood from the twistor approach.[12] For this purpose, let us define

$$G_{\rm A}(\xi) = \frac{1}{(2\pi)^2} \sum_{(k_\Lambda)_+} n_{k_\Lambda} \mathrm{Li}_2\left(e^{-2\pi \mathrm{i} k_\Lambda \xi^\Lambda}\right)\,, \tag{4.5}$$

[12] A contour integral presentation of (4.1) of the form (2.29) was given in [34], but its twistorial interpretation is obscured by issues of convergence.

where the sum runs over the set (here $H_2^-(Y) = -H_2^+(Y)$)

$$(k_\Lambda)_+ \equiv \{k_0 \in \mathbb{Z}, \quad k_a\gamma^a \in H_2^+(Y) \cup H_2^-(Y) \cup \{0\}\,, \quad \mathrm{Re}\,\left(k_\Lambda z^\Lambda\right) > 0\}\ . \tag{4.6}$$

This sum (4.5) depends on the value of the coordinates z^Λ on $\mathcal{M}$ through the last condition in (4.6), and through the coefficients n_{k_Λ}; the latter are locally constant away from the "lines of marginal stability" (LMS) where $\mathrm{Re}\,(k_\Lambda z^\Lambda)$ vanishes for a certain vector k_Λ, but may change across the LMS. In order to reproduce (4.3) in the region connected to the infinite volume limit, we require

$$n_{(k_0,k_a)} = n^{(0)}_{k_a} \quad \text{for} \quad k_a\gamma^a \neq 0\,, \qquad n_{(k_0,0)} = 2n^{(0)}_0 = -\chi_Y\,. \tag{4.7}$$

Then, to the three-patch covering and transition functions (3.2) describing the perturbative moduli space, we add two additional transition functions

$$H^{[0\ell_+]} = -\frac{\mathrm{i}}{2}\,G_{\mathrm{A}}(\xi)\,, \qquad H^{[0\ell_-]} = -\frac{\mathrm{i}}{2}\,\bar{G}_{\mathrm{A}}(\xi)\,, \tag{4.8}$$

which are associated with *open* contours extending from $\mathbf{z}=0$ to $\mathbf{z}=\infty$ along the semi-infinite imaginary axes $\ell_\pm \equiv \mathrm{i}\mathbb{R}^\pm$. This construction requires the extension of our formalism to open contours, as discussed at the end of section 2.3. Using the integral representation

$$\frac{1}{4}\int_0^\infty \frac{\mathrm{d}t}{t}\left(\alpha t + \frac{\beta}{t}\right) e^{-\frac{1}{2}\left(\alpha t + \frac{\beta}{t}\right)} = \sqrt{\alpha\beta}\,K_1\left(\sqrt{\alpha\beta}\right) \tag{4.9}$$

valid for $\mathrm{Re}\,\alpha > 0$, $\mathrm{Re}\,\beta > 0$, it is then easy to see that (2.29) precisely reproduces the contact potential (4.3). A more complete analysis of the structure of the twistor space will be given after we incorporate the B-type D-instantons.

Note that the one-loop string correction, which follows here from the non-vanishing anomalous dimension c_α, can be obtained alternatively by adding the term $k_\Lambda = 0$ to the sum in (4.5). Since this constant term does not vanish at the ends of the contour, as mentioned in the end of section 2.3, the contact potential (2.29) receives additional contributions given in (2.36) where $j = \ell_\pm$. Taking $n_{(0,0)} = -\chi_Y/2$, it is easy to check that these contributions reproduce the one-loop term in (4.3).

4.2 Covariantizing under electric-magnetic duality

In general, instanton corrections break all continuous isometries of $\mathcal{M}$ to a discrete subgroup. Thus, hypermultiplets can no longer be dualized to tensor multiplets, and the projective superspace description in terms of the $\mathcal{O}(2)$ multiplets breaks down. However, as explained in section 2, linear perturbations of toric QK manifolds can still be described by a set of generating functions $H^{[ij]}_{(1)}$, which now depend on all complex coordinates ξ^Λ, $\tilde{\xi}_\Lambda$ and α on $\mathcal{Z}$.

In the case of D-instantons, i.e. Euclidean D-branes wrapping arbitrary cycles in $H_3(X,\mathbb{Z})$ (in Type IIA string theory) or $H_{\mathrm{even}}(Y,\mathbb{Z})$ (in Type IIB string theory), the translational isometry along the NS-axion is preserved, and one should therefore restrict to perturbations which are independent of $\alpha^{[j]}$. As explained in section 2.3, this considerably simplifies the analysis, since e.g. the geometry of $\mathcal{Z}$ can be described by a single contact potential, as in the unperturbed case. Moreover, while we may in principle perturb of the A-instanton corrected toric geometry, we choose to treat *all* instantons as perturbations around the perturbative geometry described by the transition functions given in (3.2).

Using the action of electric-magnetic duality described in section 3.1, the function (4.5) describing the D-instanton part may be covariantized into

$$G_{\mathrm{A/B}}(\xi^\Lambda,\rho_\Lambda)=\frac{1}{(2\pi)^2}\sum_{(\gamma)_+}n_\gamma\,\mathrm{Li}_2\left(e^{2\pi\mathrm{i}(l^\Lambda\rho_\Lambda-k_\Lambda\xi^\Lambda)}\right)\,, \tag{4.10}$$

and used as a replacement for G_{A} in the transition functions (4.8), which we now treat as infinitesimal perturbations:[13]

$$H^{[0\ell+]}_{(1)}=-\frac{\mathrm{i}}{2}\,G_{\mathrm{A/B}}(\xi^\Lambda,\rho_\Lambda)\,,\qquad H^{[0\ell-]}_{(1)}=-\frac{\mathrm{i}}{2}\,\bar G_{\mathrm{A/B}}(\xi^\Lambda,\rho_\Lambda)\,. \tag{4.11}$$

The precise range of summation $(\gamma)_+$ in (4.10) is left unspecified at this stage; it must however have support on charges $\gamma=(k_\Lambda,l^\Lambda)$ with $\mathrm{Re}\,(W_\gamma)>0$, where

$$W_\gamma\equiv\mathcal{R}\left(k_\Lambda z^\Lambda-l^\Lambda F_\Lambda(z)\right)\,, \tag{4.12}$$

and reproduce (4.6) when $l^\Lambda=0$. There may be additional restrictions on the charge vector γ generalizing the effective or anti-effective condition in (4.6), but we shall leave this question open. It is also important to note that (4.10) was obtained by covariantizing the contributions of instantons with $l^\Lambda=0$, which have a vanishing Hitchin functional; it is logically possible that the sum (4.17) may include only states related to those states by electric-magnetic duality, in particular with a vanishing Hitchin functional too. At any rate, eqs. (4.10) and (4.8) parametrize the most general QK perturbation of the one-loop corrected metric, consistent with integer shifts of the RR moduli $\zeta^\Lambda,\tilde\zeta_\Lambda$ and continuous shifts of the NS axion σ.

Using the general results from section 2.3, it is straightforward albeit tedious to compute the contact twistor lines and contact potential, to first order in the perturbation (4.10). Using (2.33) and (3.5), we find that the perturbed twistor lines in the patch $\hat{\mathcal{U}}_0$, most appropriate for assessing symplectic invariance, are given by

$$\xi^\Lambda=\zeta^\Lambda+\mathcal{R}\left(\mathbf{z}^{-1}z^\Lambda-\mathbf{z}\,\bar z^\Lambda\right)+\frac{1}{16\pi^2}\sum_\gamma n_\gamma\,l^\Lambda\,\mathcal{I}^{(1)}_\gamma(\mathbf{z})\,, \tag{4.13a}$$

$$\rho_\Lambda=\tilde\zeta_\Lambda+\mathcal{R}\left(\mathbf{z}^{-1}F_\Lambda-\mathbf{z}\,\bar F_\Lambda\right)+\frac{1}{16\pi^2}\sum_\gamma n_\gamma\,k_\Lambda\,\mathcal{I}^{(1)}_\gamma(\mathbf{z})\,, \tag{4.13b}$$

$$\begin{aligned}\tilde\alpha=\;&\sigma+\mathcal{R}(\mathbf{z}^{-1}W-\mathbf{z}\,\bar W)+\frac{\mathrm{i}\chi_X}{24\pi}\log\mathbf{z}+\frac{\mathrm{i}}{2\pi^2}\sum_\gamma n_\gamma\left(\mathbf{z}^{-1}W_\gamma+\mathbf{z}\bar W_\gamma\right)\mathcal{K}_\gamma\\&+\frac{1}{16\pi^2}\sum_\gamma n_\gamma\left[\frac{1}{\pi\mathrm{i}}\mathcal{I}^{(2)}_\gamma(\mathbf{z})+\left(\Theta_\gamma+\mathbf{z}^{-1}W_\gamma-\mathbf{z}\bar W_\gamma\right)\mathcal{I}^{(1)}_\gamma(\mathbf{z})\right]\,,\end{aligned} \tag{4.13c}$$

[13] In this equation, $H^{[0\ell\pm]}_{(1)}$ is a function of $\xi^\Lambda_{[0]}$ and $\tilde\xi^{[\ell\pm]}_\Lambda$, which are equal to ξ^Λ and $\frac{\mathrm{i}}{2}\rho_\Lambda$ at this order.

where the sum over $\gamma = (k_\Lambda, l^\Lambda)$ runs over the union of $(\gamma)_+$ and its opposite $(\gamma)_-$ (in particular, it does not include the zero class). In (4.13), W_γ and W are as defined in (3.7) and (4.12) with

$$
\begin{aligned}
\zeta^\Lambda \equiv A^\Lambda\,, \qquad \tilde\zeta_\Lambda &\equiv B_\Lambda + A^\Sigma \operatorname{Re} F_{\Lambda\Sigma} + \frac{1}{4\pi^2} \operatorname{Im} F_{\Lambda\Sigma} \sum_\gamma n_\gamma\, l^\Sigma \mathcal{K}_\gamma\,, \\
\sigma &\equiv -2B_\alpha - A^\Lambda B_\Lambda + \frac{1}{2\pi^2} A^\Lambda \operatorname{Im} F_{\Lambda\Sigma} \sum_\gamma n_\gamma\, l^\Sigma \mathcal{K}_\gamma\,, \qquad (4.14)\\
\Theta_\gamma &\equiv k_\Lambda A^\Lambda - l^\Lambda \left(B_\Lambda + A^\Sigma \operatorname{Re} F_{\Lambda\Sigma}\right) \qquad (4.15)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{K}_\gamma &\equiv \sum_{m=1}^{\infty} \frac{1}{m} \sin\left(2\pi m \Theta_\gamma\right)\, K_0\left(4\pi m |W_\gamma|\right)\,, \\
\mathcal{I}_\gamma^{(\nu)}(\mathbf{z}) &\equiv \sum_{m=1}^{\infty} \sum_{s=\pm 1} \frac{s^\nu}{m^\nu}\, e^{-2\pi \mathrm{i} s m \Theta_\gamma} \int_0^\infty \frac{\mathrm{d}t}{t} \frac{t - \epsilon_\gamma s \mathrm{i}\mathbf{z}}{t + \epsilon_\gamma s \mathrm{i}\mathbf{z}}\, e^{-2\pi m \epsilon_\gamma \left(t^{-1} W_\gamma + t \bar W_\gamma\right)}\,,
\end{aligned}
\qquad (4.16)
$$

where $\epsilon_\gamma = \operatorname{sign}(\operatorname{Re} W_\gamma)$. Eqs. (4.14) generalize the relations (3.3) to the presence of D-instanton corrections and ensure that under electric-magnetic duality, $(\zeta^\Lambda, \tilde\zeta_\Lambda)$ and $(\xi^\Lambda, \rho_\Lambda)$ transform as a vector while σ and $\tilde\alpha$ are invariant. Note that in the leading instanton approximation, $\Theta_\gamma = k_\Lambda \zeta^\Lambda - l^\Lambda \tilde\zeta_\Lambda$. The twistor lines in other patches can be obtained by applying the transformation rule (2.31).

Finally, inserting (4.10) in (2.34), we obtain the perturbed contact potential,

$$
e^{\Phi_{\mathrm{A/B}}} = \frac{\mathcal{R}^2}{4} K(z,\bar z) + \frac{\chi_X}{192\pi} + \frac{1}{8\pi^2} \sum_\gamma n_\gamma \sum_{m>0} \frac{|W_\gamma|}{m} \cos\left(2\pi m \Theta_\gamma\right) K_1(4\pi m |W_\gamma|)\,. \qquad (4.17)
$$

Through (2.6) this result encodes the Kähler potential on $\mathcal{Z}$.

In the absence of instanton corrections, (4.13), (4.14), (4.17) reduce to (3.6), (3.3), (3.4), respectively.[14] While the results above hold in general to first order in the instanton corrections, they become exact in the case $l^\Lambda = 0$, where the toric isometries are unbroken. It would be interesting to investigate the transformation properties of (4.13) under S-duality in this case. S-duality should become manifest after Poisson resummation on k_0, but will require correcting the tree-level action (3.19) and mirror map (3.20). S-duality is clearly broken by D-instanton effects, but may be recovered once NS5-brane instantons are included. Both of these issues lie beyond the scope of this work. Finally, note that the description of the twistor space given in this and the preceding section is not rigorous due to the occurrence of open contours. A more rigorous construction of the twistor space can be found in appendix A.2.

4.3 General D-instanton corrections

In this section we interpret the corrections to the contact potential (4.17) as Euclidean D-brane instantons.

[14] The apparent difference of the contact potentials by the factor of 2 is due to that the sum in (4.17) goes over all lattice of charges, including the negative ones.

Using the asymptotic behavior $K_s(z) \sim \sqrt{\pi/(2z)}e^{-z}(1+\mathcal{O}(1/z))$ of the modified Bessel function, the classical instanton action associated to a general term in the sum (4.17) with $m > 0$ and $(k_\Lambda, l^\Lambda) \neq 0$ is given by

$$S_{\rm cl} = 4\pi m |W_\gamma| + 2\pi \mathrm{i} m \Theta_\gamma \,. \tag{4.18}$$

From the point of view of Type IIA string theory compactified on the CY threefold X, instantons should correspond to Euclidean D2-branes wrapping a special Lagrangian submanifold in the homology class $\gamma = k_\Lambda \gamma^\Lambda - l^\Lambda \gamma_\Lambda \in H_3(X, \mathbb{Z})$ (or more precisely, to elements in the Fukaya category $\mathcal{F}(X)$, see e.g. [62] for a nice review). Rewriting (4.18) as

$$S_{\rm cl} = 8\pi m \sqrt{e^\phi - \frac{\chi_X}{192\pi}}\, |Z(\gamma)| + 2\pi \mathrm{i} m \left(k_\Lambda \zeta^\Lambda - l^\Lambda \tilde\zeta_\Lambda \right) \,, \tag{4.19}$$

where $Z(\gamma)$ is the normalized central charge function on $H_3(X, \mathbb{Z})$,

$$Z(\gamma) \equiv \frac{k_\Lambda z^\Lambda - l^\Lambda F_\Lambda(z)}{\sqrt{K(z,\bar z)}} \,, \tag{4.20}$$

and recalling that $e^{\phi/2} = 1/g_4$, we recognize in the weak coupling limit $e^\phi \to \infty$ the action of m Euclidean D2-branes wrapping the 3-cycle γ. The one-loop correction proportional to χ_X can be viewed as a quantum correction to the volume of Y, and was already seen for the universal hypermultiplet in [63]. The infinite series of power corrections to the exponential behavior of the modified Bessel function K_1 should correspond to perturbative corrections in the background of the D-instanton. The "instanton measure" n_γ is unknown at this stage, but presumably counts the number of states in $\mathcal{F}(X)$ with charge γ.

On the Type IIB side, BPS D-instantons correspond to elements in the derived category of coherent sheaves $\mathcal{D}(Y)$ [64, 65]. In plain (but oversimplified) terms, they are obtained by wrapping N Euclidean D5-branes on Y, and allowing a non-trivial supersymmetric U(N) gauge configuration F on their worldvolume.[15] In the large volume limit, their classical action is given by [62, 66, 67]

$$S_{\rm cl} = \tau_2 \left| \int_Y e^{-\mathcal{J}} \mathrm{ch}(F) \sqrt{\mathrm{td}(Y)} \right| + \mathrm{i} \int_Y A\, e^{-B} \mathrm{ch}(F) \sqrt{\mathrm{td}(Y)} \,, \tag{4.21}$$

where ch and td denote the Chern character and Todd class,

$$\begin{aligned} \mathrm{ch} &= c_0 + c_1 + \left(\frac{1}{2} c_1^2 - c_2 \right) + \frac{1}{2} \left(c_3 - c_1 c_2 + \frac{1}{3} c_1^3 \right) + \dots \,, \\ \mathrm{td} &= 1 + \frac{1}{2} c_1 + \frac{1}{12} (c_2 + c_1^2) + \frac{1}{24} c_1 c_2 + \dots \end{aligned} \tag{4.22}$$

with $c_1(Y) = 0$ by the CY condition, and A is the sum of RR forms,

$$A = A^{(0)} + A^{(2)} + A^{(4)} + A^{(6)} \,. \tag{4.23}$$

[15] Instantons with zero D5-brane charge can be obtained as bound states of D5 and anti-D5-branes, which are also in $\mathcal{D}(Y)$.

The actions (4.21) and (4.18) match in the large volume limit, provided the charges and RR scalars are identified via

$$\begin{aligned} \mathrm{ch}(F)\sqrt{\mathrm{td}(Y)} &= l^0 + l^a\,\omega_a - k_a\,\omega^a + k_0\,\omega_Y\,, && (4.24a)\\ A\,e^{-B} &= \zeta^0 - \zeta^a\,\omega_a - \tilde\zeta_a\,\omega^a - \tilde\zeta_0\,\omega_Y\,. && (4.24b)\end{aligned}$$

Eq. (4.24a) gives the standard relation between charges and the characteristic classes of F,

$$\begin{aligned} &l^0 = N\,, \qquad l^a = \int_{\gamma^a} c_1(F)\,, \qquad k_a = -\int_{\gamma_a}\left[\left(\frac{1}{2}\,c_1^2(F) - c_2(F)\right) + \frac{N}{24}\,c_2(Y)\right],\\ &k_0 = \int_Y \left[\frac{1}{2}\left(c_3(F) - c_1(F)c_2(F) + \frac{1}{3}\,c_1^3(F)\right) + \frac{1}{24}\,c_2(Y)c_1(F)\right], \end{aligned} \tag{4.25}$$

while (4.24b) combined with (3.11) reproduces the mirror map (3.20).

As in the Type IIA case, power corrections to the exponential behavior of K_1 should correspond to perturbative corrections in the instanton background, and the "instanton measure" n_γ should correspond to the number of states in $\mathcal{D}(Y)$ with D$(-1,1,3,5)$ charges (k_0, k_a, l^a, l^0) in $H_{\rm even}(Y)$, possibly with a restriction on the allowed charges.

4.4 Exact twistor space in presence of D-instantons

We now suggest a construction of the twistor space $\mathcal{Z}$ in presence of D-instanton corrections, essentially identical to the one given in [38] in the gauge theory context, which should be exact in the absence of NS5-brane instantons.

As in [38], each charge vector $\gamma = (k_\Lambda, l^\Lambda)$ defines a pair of "BPS rays" $\ell_\pm(\gamma)$ on $\mathbb{C}P^1$ and two hemispheres $V_\pm(\gamma)$ defined by

$$\ell_\pm(\gamma) = \{\mathbf{z}:\ \pm W_\gamma/\mathbf{z} \in \mathrm{i}\mathbb{R}^-\}\,, \qquad V_\pm(\gamma) = \{\mathbf{z}:\ \pm\,\mathrm{Im}\,(W_\gamma/\mathbf{z}) < 0\}\,, \tag{4.26}$$

in such a way that $|e^{\mp\mathrm{i}(k_\Lambda\xi^\Lambda - l^\Lambda\rho_\Lambda)}| < 1$ in $V_\pm(\gamma)$, and that $e^{\mp\mathrm{i}(k_\Lambda\xi^\Lambda - l^\Lambda\rho_\Lambda)}$ is exponentially suppressed at $\mathbf{z}\to 0$ and $\mathbf{z}\to\infty$ in $V_\pm$. We propose that across all BPS rays $\ell_\pm(\gamma)$ the complex contact structure experiences finite contact transformations U_γ generated by

$$S_\gamma^{[ij]}(\xi^\Lambda_{[i]}, \tilde\xi_\Lambda^{[j]}, \alpha^{[j]}) = \alpha^{[j]} + \xi^\Lambda_{[i]}\,\tilde\xi_\Lambda^{[j]} + \frac{\mathrm{i}}{2(2\pi)^2}\,n_\gamma\,\mathrm{Li}_2\left(e^{\mp 2\pi\mathrm{i}(k_\Lambda\xi^\Lambda_{[i]} + 2\mathrm{i}l^\Lambda\tilde\xi_\Lambda^{[j]})}\right). \tag{4.27}$$

Actually, the precise angular location of the BPS rays $\ell_\pm(\gamma)$ where the contact transformation is performed is unimportant, provided they stay inside the hemispheres $V_\pm(\gamma)$, respectively, and the angular order between BPS rays of different charges is preserved. Thus, one may "pile up" the BPS rays up in just two composite rays located on the positive and negative imaginary axis [38]. Across these two rays, the contact structure experiences the product of all elementary contact transformations (4.27), ordered counterclockwise according to the phase of the central charge W_γ:

$$U_+ = \prod^{\curvearrowleft}_{\mathrm{Re}\,(W_\gamma)>0} U_\gamma\,, \qquad U_- = \prod^{\curvearrowleft}_{\mathrm{Re}\,(W_\gamma)<0} U_\gamma\,, \tag{4.28}$$

where the product denotes the composition of contact transformations. The latter can be computed from the generating functions using (2.13). Together with the contact transformations (3.2) determining the perturbative part of the hypermultiplet metric, this defines a twistor space $\mathcal{Z}$ which should provide the exact metric on the hypermultiplet moduli space $\mathcal{M}$ in the absence of NS5-brane instantons.

The ordering of the BPS rays depends on the moduli z^Λ via the central charge function (4.20), and changes across lines of marginal stability (LMS) where the phase of the central charges of two BPS instantons γ_1 and γ_2 become aligned. At the same time, the value of the invariants n_γ is expected to change, in such a way that the products S_+ and S_- stay invariant. As explained in [38] and further discussed in section 5.2, this consistency condition is identical in form to the wall-crossing formula for generalized Donaldson-Thomas invariants found in [39]. Thus, it strongly suggests that the instanton measure should be identified to these generalized Donaldson-Thomas invariants.

In the leading instanton approximation, the infinite products in (4.28) reduce to an infinite sum, and the contact transformations $S_\pm$ are generated by the functions $G_{\mathrm{A/B}}$ and $\bar{G}_{\mathrm{A/B}}$ as described in section 4.2.

5 Discussion

In this work, we have studied D-instanton corrections to the hypermultiplet branch $\mathcal{M}$ of Type II compactifications on a Calabi-Yau threefold using twistor techniques. Our main result is the instanton-corrected "contact potential" (4.17), which, together with the "contact twistor lines" (4.13) provides sufficient information to determine the instanton-corrected QK metric on the hypermultiplet moduli space, in the "leading instanton" approximation. These results follow from a simple deformation of the complex contact geometry on the twistor space $\mathcal{Z}$ of $\mathcal{M}$, controlled by the holomorphic function (4.10) (more accurately, a section of $H^1(\mathcal{Z}, \mathcal{O}(2))$. In section 4.4, we have proposed how this perturbation could be elevated to a finite deformation of the twistor space $\mathcal{Z}$, which should yield the exact QK metric on $\mathcal{Z}$ in the sector without NS5 branes. In the remainder of this work, we comment on some possible relations of these results to the counting of 4D BPS black holes, and to the wall-crossing formula of Kontsevich and Soibelman, and speculate on the form of the NS5-brane instanton corrections.

5.1 Instanton corrections and black hole partition functions

In [6], it was suggested on general ground that instanton-corrected BPS couplings in three dimensions may provide a useful packaging for the BPS black hole degeneracies in four dimensions. Here, we apply these general ideas to the case of $\mathcal{N} = 2$ supersymmetry, and argue that the instanton measure n_γ in Type IIB (resp. Type IIA) string theory compactified on Y is directly related to the microscopic indexed degeneracy of 4D black holes in Type IIA (resp. Type IIB) string theory compactified on the *same* Calabi-Yau threefold Y.

For this purpose, consider the compactification of Type IIB string theory down to three dimensions on the product of Y times a circle of radius $R = e^U l_P$, where l_P is the

4D Planck length. The moduli space in three dimensions factorizes into the product

$$\mathcal{M}_3 = \mathcal{M}^B_{\rm HM} \times \mathcal{M}^B_{\rm VM} \tag{5.1}$$

of two QK manifolds, of dimension $4(h^{1,1}(Y)+1)$ and $4(h^{1,2}(Y)+1)$, respectively.

The first factor $\mathcal{M}^B_{\rm HM}$ is independent of the radius R (since vector multiplets and neutral hypermultiplets are decoupled at two-derivative order), and coincides with the hypermultiplet moduli space $\mathcal{M}^B_{\rm HM}$ in four dimensions. The latter was described at the perturbative level in section 3.2, and receives instanton corrections from Euclidean D-branes wrapping supersymmetric cycles in $H_{\rm even}(Y)$ as found in section 4.3.

On the other hand, the second factor $\mathcal{M}^B_{\rm VM}$ contains the radius R/l_P, the complex structure of Y, the electric and magnetic Wilson lines $(\tilde\zeta_\Lambda, \zeta^\Lambda)$ of the graviphoton and vector multiplets in $D=4$, and the NUT scalar σ (dual to the off-diagonal part of the metric). In the limit $R \gg l_P$, it is given by the c-map of the complex structure moduli space $\mathcal{M}_{\rm cs}(Y)$. At finite R, $\mathcal{M}^B_{\rm VM}$ is expected to receive loop corrections from Kaluza-Klein states running around the Euclidean circle, and instanton corrections from 4D BPS black holes of charge $\gamma = (p^\Lambda, q_\Lambda)$ whose worldline winds around the circle.[16] The classical action of these configurations is given by the mass of 4D black hole times the length of the circle, plus the coupling to the Wilson lines,

$$S_{\rm cl} = 2\pi e^U |Z(\gamma)| + 2\pi {\rm i}(\zeta^\Lambda q_\Lambda - \tilde\zeta_\Lambda p^\Lambda)\,, \tag{5.2}$$

where the central charge $Z(\gamma)$ (a function of the vector multiplet moduli and the black hole charges) takes the same form as in (4.20). In addition to these σ-independent contributions, there are also Euclidean configurations with NUT charge $k \neq 0$, inducing terms proportional to $e^{{\rm i}k\sigma}$ in the low-energy effective action. Similarly, in Type IIA compactified on $Y \times S^1$ the moduli space takes the product form $\mathcal{M}_3 = \mathcal{M}^A_{\rm VM} \times \mathcal{M}^A_{\rm HM}$, with the role of Kähler and complex structure moduli being exchanged.

It is well-known that T-duality along the circle exchanges the two factors in the three-dimensional moduli space (5.1) [70],

$$\mathcal{M}^B_{\rm HM} = \mathcal{M}^A_{\rm VM}\,, \qquad \mathcal{M}^B_{\rm VM} = \mathcal{M}^A_{\rm HM}\,, \tag{5.3}$$

in particular it exchanges the radius U with the four-dimensional dilaton ϕ. This implies that (i) the one-loop correction on the hypermultiplet branch, proportional to χ_Y, should reproduce the loop corrections from KK states on the vector multiplet branch, and (ii) the D-instanton contributions to $\mathcal{M}^B_{\rm HM}$ (already present in $D=4$) should be mapped to black hole instantons contributions to $\mathcal{M}^A_{\rm VM}$ (arising in the compactification to $D=3$). While we have not attempted to check (i), it is clear that the classical actions (4.19) and (5.2) agree provided T-duality exchanges

$$e^U \leftrightarrow 4\sqrt{e^\phi - \frac{\chi_X}{192\pi}}\,, \tag{5.4}$$

[16] Analogous contributions of 4D monopoles to the 3D effective potential are famously responsible for the confinement of $D=2+1$ compact Maxwell theory [68]. The reason that only BPS black holes can contribute to the metric is the standard saturation of fermionic zero-modes, see e.g. [69].

implying a one-loop correction to the usual c-map.

At this point, it may be worthwhile to note that the correction terms in the contact potential (4.17) are identical in form to the radial wave function for BPS black holes computed in [52, 71], except for the one-loop correction proportional to χ_X. This is hardly surprising, since in the context of $D = 4, \mathcal{N} = 2$ supergravity, spherically symmetric BPS instanton configurations are described by the same geodesic motion which controls the radial profile of BPS black holes [6, 72]. The radial quantization of BPS black hole solutions leads to a quantum Hilbert space of functions which happens to coincide with the space $H^1(\mathcal{Z}, \mathcal{O}(2))$ of QK deformations of $\mathcal{M}$. This has an important practical consequence: for a fixed value of the moduli z^Λ, the instanton configurations which dominate the sum (4.17) are given by extremizing the central charge $|Z(\gamma)|$ with respect to the charges γ. It would be interesting to investigate this "reverse attractor mechanism" further.

While the action of D-instantons and black holes are easily matched, the relation between the summation measures is more subtle. In discussing this issue, it is useful to bear in mind an analogous but simpler problem, namely the relation between the D(−1)-instanton measure for R^4 couplings in $D = 10$ Type IIB string theory [58], and the Witten index of D0-branes in $D = 10$ Type IIA string theory [73]. Since N D0-branes have a single bound state at threshold for any $N > 0$, the index should be equal to 1. The Witten index is given by a functional integral in U(N) supersymmetric quantum mechanics with 16 supercharges, with periodic boundary conditions for the fermions along the Euclidean time circle of length β. However, due to flat directions in the potential, only the low temperature limit $\beta \to \infty$ is expected to yield the Witten index $\Omega(N)$. On the other hand, the D(−1)-instanton measure $\mu(N)$ is given by a U(N) matrix integral, i.e. the reduction of the quantum mechanics on a circle of vanishing size $\beta \to 0$. After regulating volume divergences, the difference

$$\Omega(N) - \mu(N) = \int_0^\infty \mathrm{d}\beta \, \frac{\partial}{\partial\beta} \mathrm{Tr}\left[(-1)^F e^{-\beta H}\right] , \tag{5.5}$$

rewritten as a "bulk" contribution to the index [74, 75], was evaluated in [73] and found to agree with the answer predicted by S-duality [58], $\mu(N) = \sum_{d|N} 1/d^2$, $\Omega(N) = 1$. In particular, when N is a prime number, the instanton measure and the Witten index agree.

This analogy suggests that in the absence of marginal directions in the potential, i.e. for non-threshhold bound states, the instanton measure n_γ for Type IIB/Y and the indexed degeneracy $\Omega(k_\Lambda, l^\Lambda)$ of 4D BPS black holes in Type IIA/Y should agree (with a similar statement upon exchanging Type IIA and IIB). Thus, the metric on the hypermultiplet branch appears to be a very convenient packaging for the indexed degeneracies of 4D BPS black hole in the dual theory. In particular, it gives a natural way to encode the dependence of the black hole spectrum on the values of the moduli at spatial infinity, as we now discuss.

5.2 Wall crossing

The spectrum of single-particle states is known to jump across lines of marginal stability (LMS), where the phase of the central charge of two BPS states align. This phenomenon has been much studied in the context of $\mathcal{N} = 2$ supersymmetric gauge theories (see e.g. [76–78]),

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and also takes place in $\mathcal{N}=2$ supergravity theories, where it has a macroscopic description in terms of multi-centered black hole configurations [79]: as the LMS is approached from one side, the distance between the centers diverges and the configuration becomes unbound. On the other side of the LMS, the bound state no longer exists as a single-particle state, but it is replaced by a continuum of multi-particle states with the same total charge. Thus, by analyzing the leading instanton contributions at a given point on the 3D moduli space in the large radius limit $U \to \infty$, one should be able to determine the BPS spectrum at that particular point. Moreover, since no massless state typically occurs on the LMS, the hypermultiplet metric is expected to be smooth, with the single instanton contribution on one side of the LMS matching the multi-instanton contribution on the other side. This should provide strong constraints on the discontinuity of the one-particle BPS spectrum across the LMS.

This idea was demonstrated recently in the context of rigidly supersymmetric gauge theories with 8 supercharges in 4 dimensions [38]. In particular, the authors showed that the hyperkähler moduli space $\mathcal{S}$ of the gauge theory compactified down to three dimensions gives a natural physical setting for the Kontsevich-Soibelman wall-crossing formula [39, 40]: the latter ensures that the leading instanton effects on the twistor space combine with each other consistently so as to produce a regular HK manifold. To see why this may be true, recall that on very general ground, the "generalized Donaldson-Thomas invariants" $\Omega(\gamma)$ must satisfy [38–40]

$$\prod_{\substack{\gamma=n\gamma_1+m\gamma_2\\ m>0,n>0}}^{\curvearrowleft} U_\gamma^{\Omega^-(\gamma)} = \prod_{\substack{\gamma=n\gamma_1+m\gamma_2\\ m>0,n>0}}^{\curvearrowright} U_\gamma^{\Omega^+(\gamma)}\,, \tag{5.6}$$

where $\Omega^-(\gamma)$ and $\Omega^+(\gamma)$ denote the value of $\Omega(\gamma)$ on either side of the LMS where the phases of $Z(\gamma_1)$ and $Z(\gamma_2)$, align. Here

$$U_\gamma \equiv \exp\left(\sum_{n=1}^{\infty} \frac{1}{n^2}\, e_{n\gamma}\right)\,, \tag{5.7}$$

where $e_\gamma \equiv e_{p,q}$ are generators of the Lie algebra

$$\left[e_{p,q}, e_{p',q'}\right] = (-1)^{p^\Lambda q'_\Lambda - p'^\Lambda q_\Lambda}\, \left(p^\Lambda q'_\Lambda - p'^\Lambda q_\Lambda\right)\, e_{p+p',q+q'}\,. \tag{5.8}$$

Except for the sign $(-1)^{p^\Lambda q'_\Lambda - p'^\Lambda q_\Lambda}$, which can be absorbed into a redefinition of $e_{p,q}$ by a choice of "quadratic refinement" [38], this is the algebra of infinitesimal symplectomorphisms on the complex torus $(\mathbb{C}^\times)^{2n}$, where $e_{p,q} = e^{2\pi \mathrm{i}(q_\Lambda \xi^\Lambda - p^\Lambda \tilde\xi_\Lambda)}$ is a basis of contact Hamiltonians and the commutator is the Poisson bracket $[\mu_1, \mu_2] = (\mathrm{i}/2\pi)(\partial_{\xi^\Lambda}\mu_1 \partial_{\tilde\xi_\Lambda}\mu_2 - \partial_{\xi^\Lambda}\mu_2 \partial_{\tilde\xi_\Lambda}\mu_1)$. Indeed, this complexified torus can be identified as the twistor space $\mathcal{Z}_\mathcal{S}$ of the HK manifold $\mathcal{S}$, and the relation (5.8) guarantees the consistency of the symplectic structure across the LMS [38].

Returning to the case of Type IIB string theory compactified on Y, where the moduli space $\mathcal{M}$ is QK rather than HK, it is natural to expect that a similar construction operates at the level of the twistor space $\mathcal{Z}$ equipped with its complex contact structure. Indeed,

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by using the requirement of S-duality invariance, we have found that in the "leading instanton" approximation, instanton corrections induce contact transformations generated by the sum of dilogarithms (4.10). The latter originated by Poisson resummation from the trilogarithms present in the worldsheet instanton sum (3.13). The occurrence of the same dilogarithm function in (4.10) as in (5.7) gives a strong hint that the instanton measure n_γ should be identified with the generalized Donaldson-Thomas invariants of [39], and in turn with the index degeneracies of 4D BPS black holes.

Alas, this sequence of identifications raises serious puzzles: the indexed degeneracies $\Omega(\gamma)$ of large black holes are known to grow exponentially as $\Omega \sim e^{\lambda^2}$ when the charge vector γ is rescaled by a common factor λ, while the exponential of the classical action decreasing only as $e^{-\lambda}$. Assuming that the instanton measure were equal to the black hole degeneracy, it would seem impossible that the instanton sum could converge at all.[17] It is conceivable however that the instanton measure, and indeed the generalized Donaldson-Thomas invariants, may have support on "polar" states [80] (i.e. states with imaginary entropy in the supergravity approximation), whose degeneracies grow less rapidly. Another puzzle is the absence of quantum corrections to the hypermultiplet moduli space metric in Type II compactifications on certain self-mirror CY manifolds [81], which are nevertheless expected to have a non-trivial spectrum of BPS black holes. It would be interesting to check whether black holes or instantons in these models have accidental fermionic zero-modes which forbid their contribution to the index and/or to the metric.

5.3 NS5-brane instantons

We now briefly comment on the effects of NS5-branes on the hypermultiplet metric.[18] Since these instanton configurations carry magnetic charge under the NS two-form B, they must break the shift isometry along the NS axion direction to a discrete subgroup. Since the NS axion enters linearly in the complex coordinate $\tilde\alpha = 4\mathrm{i}\alpha^{[0]} + 2\mathrm{i}\tilde\xi^{[0]}_\Lambda \xi^\Lambda_{[0]} = \sigma + \dots$, such corrections must take the form, at the infinitesimal level,

$$H^{[ij]}_{(1)}(\xi^\Lambda_{[i]}, \tilde\xi^{[j]}_\Lambda, \alpha^{[j]}) \sim \exp\left(-4k\,\alpha^{[j]}\right) H^{[ij]}_{(1),k}(\xi^\Lambda_{[i]}, \tilde\xi^{[j]}_\Lambda)\,, \tag{5.9}$$

where k is the NS5-brane charge. This causes some technical difficulty, since $H^{[ij]}_{(1)}$ is no longer independent of $\alpha^{[j]}$, and the QK geometry can no longer be described by a single $\mathbf{z}$-independent contact potential.

A more conceptual problem however is the fact that for non-vanishing NS5-brane charge k, the translations along the RR axionic directions no longer commute. Instead, they generate a Heisenberg algebra

$$\left[P^\Lambda, Q_\Sigma\right] = -2\delta^\Lambda_\Sigma K\,, \tag{5.10}$$

where

$$P^\Lambda = \partial_{\tilde\zeta_\Lambda} - \zeta^\Lambda\partial_\sigma\,, \qquad Q_\Lambda = -\partial_{\zeta^\Lambda} - \tilde\zeta_\Lambda\partial_\sigma\,, \qquad K = \partial_\sigma\,, \tag{5.11}$$

[17] For similar reasons, the 6-derivative BPS couplings in 3D string vacua with 16 supercharges, or for 14-derivative couplings in 3D vacua with 32 supercharges, may be ill defined.

[18] These are dual to the effects of Euclidean configurations with non-zero NUT charge on the vector multiplet branch in 3 dimensions mentioned below eq. (5.2).

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with the effective Planck constant K being proportional to the NS5-brane charge k. Thus, a Fourier decomposition such as (4.10) is no longer applicable. Instead, the plane wave solutions appearing in (4.10) should be replaced by wave functions of a charged particle on a torus with magnetic flux $k\mathrm{d}\xi^\Lambda \wedge \mathrm{d}\rho_\Lambda$. It is tempting to speculate that the coefficients of this non-Abelian Fourier decomposition may be related to the "quantum invariants" defined in [39], with the classical dilogarithm Li_2 being replaced by its quantum version.

In the absence of an obvious guess for the form of these NS5-corrections, one may consider the longer route proposed in [34]: by mirror symmetry, the B-type D2-brane instantons in Type IIA are mapped to D3- and D5-brane instantons. A further use of S-duality in principle would map D5-brane instantons to NS5-brane instantons in Type IIB, and finally to NS5-brane instantons in Type IIA via mirror symmetry. Given the complexity of the transformation rules (3.19) of the twistor lines at tree-level, it is a challenging problem to covariantize the B-type instanton contributions under S-duality. We hope to return to this issue in a forthcoming publication.

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A More twistor constructions

In this appendix, we revisit the twistor space formulation of the moduli space $\mathcal{S}$ underlying $\mathcal{N} = 2$ gauge theories on $\mathbb{R}^3 \times S^1$ in [38], focusing on the case of SU(2) gauge group without matter for simplicity. The hyperkähler metric on $\mathcal{S}$ resulting from integrating out one BPS particle of electric charge $q > 0$ winding around the circle was constructed in [35, 70]. In section A.1, we construct a set of local coordinates on its twistor space $\mathcal{Z}_\mathcal{S}$ which cover the whole $\mathbb{C}P^1$, including the north and south pole where the coordinates introduced in [38] have an essential singularity. This is important since in the absence of such a covering, it is not clear (to us) why the holomorphic symplectic form $\Omega(\zeta)$ should be a global $\mathcal{O}(2)$ section.

As it turns out, the symplectomorphisms underlying this construction are essentially identical to the contact transformations which determine the instanton-corrected twistor space for the hypermultiplet branch discussed in section 4. In section A.2, we use this observation to provide a rigorous construction of the twistor space of the hypermultiplet moduli space in Type II compactifications in the leading instanton approximation.

A.1 The Ooguri-Vafa metric revisited

The authors of [38] parametrize the real twistor lines on the twistor space $\mathcal{Z}_\mathcal{S}$ of $\mathcal{S}$ by

$$\begin{aligned}\xi(\zeta) &= \theta_e - \mathrm{i}\pi R(a/\zeta + \zeta\bar{a})\,,\\ \tilde{\xi}(\zeta) &= \theta_m - \mathrm{i}\pi R(F_a/\zeta + \zeta\bar{F}_{\bar{a}}) + \delta\tilde{\xi}\,,\end{aligned} \tag{A.1}$$

where $\xi, \tilde{\xi}, \zeta$ are complex coordinates on $\mathcal{Z}_\mathcal{S}$ such that the complex symplectic form takes the Darboux form $\Omega = \mathrm{d}\xi \wedge \mathrm{d}\tilde{\xi}$ (up to an overall normalization), and $a, \bar{a}, \theta_e, \theta_m$ are coordinates on $\mathcal{S}$. Here,

$$F_a \equiv \partial_a F\,, \qquad F(a) \equiv \frac{q^2}{4\pi\mathrm{i}}\left(a^2 \log\frac{a}{\Lambda} - \frac{3}{2}a^2\right)\,, \tag{A.2}$$

where Λ is the QCD scale, and

$$\delta\tilde{\xi} = \frac{q}{2\pi}(\mathcal{I}_+ - \mathcal{I}_-)\,, \qquad \mathcal{I}_\pm = \frac{1}{2}\int_{\ell_\pm} \frac{\mathrm{d}\zeta'}{\zeta'}\frac{\zeta'+\zeta}{\zeta'-\zeta}\log\left(1 - e^{\pm\mathrm{i}q\xi(\zeta')}\right)\,, \tag{A.3}$$

whereas $\ell_\pm$ are semi-infinite lines from 0 to ∞, lying in the middle of the half planes $V_\pm$,

$$\ell_\pm = \{\zeta :\, \pm a/\zeta \in \mathbb{R}^-\}\,, \qquad V_\pm = \{\zeta :\, \pm\,\mathrm{Re}\,(a/\zeta) < 0\}\,. \tag{A.4}$$

These conditions guarantee that $|e^{\pm\mathrm{i}q\xi(\zeta)}| < 1$ in $V_\pm$, and that $e^{\pm\mathrm{i}q\xi(\zeta)}$ is exponentially suppressed at $\zeta \to 0$ and $\zeta \to \infty$ in $V_\pm$. On the axis separating V_+ and V_-, $e^{\mathrm{i}q\xi(\zeta)}$ has modulus one, and equals one for infinitely many values of ζ accumulating near $\zeta = 0$ and ∞.

Analytic properties of complex coordinates. Our aim is to provide a regular set of coordinates on the twistor space defined above. The coordinates (A.1) do not cover the whole $\mathbb{C}P^1$, since the expression (A.3) is analytic only away from the contours of integration $\ell_\pm$. Instead, (A.1) defines two different functions $\delta\tilde{\xi}_>$ and $\delta\tilde{\xi}_<$ analytic on the half plane $V_>$ and $V_<$, respectively, with

$$V_> = \{\zeta :\ \mathrm{Im}\,(a/\zeta) > 0\}\,, \qquad V_< = \{\zeta :\ \mathrm{Im}\,(a/\zeta) < 0\}\,. \tag{A.5}$$

Moreover, under the map $\zeta \mapsto a/(\bar{a}\zeta)$, which exchanges these two regions,

$$\mathcal{I}_\pm(\zeta) + \mathcal{I}_\pm\left(\frac{a}{\bar{a}\,\zeta}\right) = 0\,, \qquad \delta\tilde{\xi}_>(\zeta) + \delta\tilde{\xi}_<\left(\frac{a}{\bar{a}\,\zeta}\right) = 0\,. \tag{A.6}$$

Using the invariance of $e^{\mathrm{i}q\xi(\zeta)}$ under this map, we may also rewrite (A.3) as an integral over the variable ξ,

$$\mathcal{I}_\pm = \int_{\pm\mathrm{i}\infty}^{\xi(\mp\sqrt{a/\bar{a}})} \frac{\mathrm{d}\xi'}{\xi'-\xi}\left[\frac{\zeta\partial_\zeta\xi}{\zeta'\partial_{\zeta'}\xi'}\right]\log\left(1 - e^{\pm\mathrm{i}q\xi}\right)\,, \tag{A.7}$$

where the factor in the square brackets is understood as a function of ξ and ξ'. Starting from (A.3), a direct computation establishes the partial differential equations

$$\left(\partial_a\partial_{\bar{a}} + \pi^2R^2\partial^2_{\theta_e}\right)\mathcal{I}_\pm = 0\,, \tag{A.8a}$$

$$\left(a\partial_a - \bar{a}\partial_{\bar{a}} + \zeta\partial_\zeta\right)\mathcal{I}_\pm = 0\,. \tag{A.8b}$$

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Note that any function of ξ is a solution of these equations. Moreover, any solution of the system can be written as $F(a,\bar{a},\theta_e,\zeta) = \Phi(a/\zeta,\bar{a}\zeta,\theta_e)$ where $\Phi(a,\bar{a},\theta_e)$ is a solution of (A.8a). The latter has a basis of solutions $K_0(2\pi Rqm|a|)\, e^{\mathrm{i}mq\theta_e}$ with $m \neq 0$.

The function $\delta\tilde{\xi}_>$ can be analytically continued into $V_+ \cap V_<$ across the contour ℓ_+. Similarly, $\delta\tilde{\xi}_<$ can be analytically continued into $V_- \cap V_>$ across the contour ℓ_-. On their common domain of definition, the two functions differ by the residue at $\zeta' = \zeta$ in (A.3),

$$\delta\tilde{\xi}_> - \delta\tilde{\xi}_< = \mathrm{i}q \log\left(1 - e^{\mathrm{i}q\xi}\right), \qquad \zeta \in V_+\,, \tag{A.9a}$$

$$\delta\tilde{\xi}_> - \delta\tilde{\xi}_< = \mathrm{i}q \log\left(1 - e^{-\mathrm{i}q\xi}\right), \qquad \zeta \in V_-\,. \tag{A.9b}$$

In particular, starting from the lower left quadrant $V_+ \cap V_>$, one may analytically continue $\delta\tilde{\xi}_>$ across ℓ_+ into the upper left quadrant $V_+ \cap V_<$, picking (A.9a), then into the upper right quadrant $V_- \cap V_<$, picking an additive constant $2\pi q m_<$, $m_< \in \mathbb{Z}$, then into the lower right quadrant $V_- \cap V_>$ across ℓ_-, picking (A.9b), and back again to the lower left quadrant, picking an extra additive constant $2\pi q m_>$, $m_> \in \mathbb{Z}$. Using the fact that

$$\mathrm{i}q \log\left(1 - e^{-\mathrm{i}q\xi}\right) - \mathrm{i}q \log\left(1 - e^{\mathrm{i}q\xi}\right) = q^2\xi \qquad \mathrm{mod}\ 2\pi q\,, \tag{A.10}$$

we conclude that the monodromy of $\delta\tilde{\xi}_>$ around $\zeta = 0$ is given by

$$\delta\tilde{\xi}_>(\zeta e^{-2\pi\mathrm{i}}) = \delta\tilde{\xi}_>(\zeta) + q^2\xi \qquad \mathrm{mod}\ 2\pi q\,. \tag{A.11}$$

Despite the fact that the part of the monodromy linear in ξ can be canceled by adding $\frac{q^2}{2\pi\mathrm{i}}\,\xi \log \xi$ to $\delta\tilde{\xi}_>$, this does not give a regular function near $\zeta = 0$ because the resulting combination still has an essential singularity at this point.

To understand how regular coordinates can be defined, let us study the behavior of (A.3) near $\zeta = 0$. Taylor expanding the rational function, Fourier expanding the logarithm and setting $\zeta' = \mp t/\bar{a}$ leads to

$$\mathcal{I}_\pm = -\int_0^\infty \frac{\mathrm{d}t}{t}\left(\frac{1}{2} + \sum_{k=1}^{\infty}\left(\mp\frac{\bar{a}\zeta}{t}\right)^k\right)\left(\sum_{m=1}^{\infty}\frac{1}{m}\, e^{\pm\mathrm{i}mq\theta_e - m\pi q R(t+|a|^2/t)}\right). \tag{A.12}$$

Exchanging the two sums with the integral over t gives

$$\mathcal{I}_\pm = -\sum_{m=1}^{\infty}\frac{1}{m}\, e^{\pm\mathrm{i}mq\theta_e}\left[K_0(2\pi qRm|a|) + 2\sum_{k=1}^{\infty}\left(\mp\zeta\sqrt{\frac{\bar{a}}{a}}\right)^k K_k(2\pi qRm|a|)\right]. \tag{A.13}$$

This Taylor series correctly reproduces the limit of $\mathcal{I}_\pm$ and of all its derivatives at $\zeta \to 0$ (irrespective of the direction of approach), in particular

$$\mathcal{I}_\pm(0) = -\sum_{m=1}^{\infty}\frac{1}{m}\, e^{\pm\mathrm{i}mq\theta_e} K_0(2\pi qRm|a|)\,. \tag{A.14}$$

However the radius of convergence in the ζ variable is zero. This reflects the existence of an essential singularity at $\zeta = 0$, and the fact that the analytical continuation of $\mathcal{I}_+$

across ℓ_+ diverges when $\zeta = 0$ is approached from V_-. In order to expose the behavior at $\zeta = 0$, one may omit the term linear in t in the exponent of (A.12) (after subtracting the ζ-independent term). The integral over t is now of Gamma function type, leading to

$$\mathcal{I}_\pm - \mathcal{I}_\pm(0) \sim -\sum_{m=1}^{\infty} \frac{1}{m}\, e^{\pm imq\theta_e} \sum_{k=1}^{\infty} \Gamma(k) \left(\mp \frac{\zeta}{m\pi qRa} \right)^k . \tag{A.15}$$

The (divergent) sum over k is recognized as the asymptotic expansion $\mathrm{e}^{-z}\,\mathrm{Ei}(z) = \sum_{k=1}^{\infty}(k-1)! z^{-k}$ valid away from the positive z axis, of the Exponential Integral $\mathrm{Ei}(z) \equiv \fint_{-\infty}^{z} \mathrm{e}^t \mathrm{d}t/t$ at $z \to \infty$. Therefore, we conclude that the analytic behavior of $\mathcal{I}_\pm$ near the origin is characterized as

$$\mathcal{I}_\pm - \mathcal{I}_\pm(0) \sim -\sum_{m=1}^{\infty} \frac{1}{m}\, e^{\pm imq\theta_e} e^{\pm m\pi qRa/\zeta}\, \mathrm{Ei}\left(\mp \frac{m\pi qRa}{\zeta} \right) . \tag{A.16}$$

An important fact is that this behavior depends on ζ, a, θ_e through the complex coordinate ξ only. Indeed, using $\xi \sim \theta_e - \mathrm{i}\pi Ra/\zeta$ at $\zeta = 0$, one finds

$$\mathcal{I}_\pm - \mathcal{I}_\pm(0) \sim -\sum_{m=1}^{\infty} \frac{1}{m}\, e^{\pm imq\xi}\, \mathrm{Ei}\left(\mp \mathrm{i}mq\xi\right) = \int_0^{\pm\mathrm{i}\infty} \frac{\mathrm{d}\xi'}{\xi - \xi'} \log\left(1 - e^{\pm \mathrm{i}q\xi'}\right) , \tag{A.17}$$

where we used the integral representation

$$e^{-xy}\, \mathrm{Ei}(xy) = \int_0^{\infty} \frac{e^{-xt}}{y - t}\, \mathrm{d}t \tag{A.18}$$

valid for $x > 0$ with identifications $x = mq$, $y = \mp\mathrm{i}\xi$, $t = \mp\mathrm{i}\xi'$, and performed the sum over m. This makes it manifest that (A.17) and (A.7) have the same discontinuity across the integration contours.

In total, these results imply the following asymptotic behavior of $\tilde{\xi}$ at $\zeta = 0$,

$$\tilde{\xi}(\zeta) \underset{\zeta\to 0}{\sim} \frac{q^2}{2\pi \mathrm{i}} \left(\xi \log \frac{\mathrm{i}\zeta\xi}{\pi R\Lambda} - \xi \right) + \frac{q}{2\pi} \sum_{m\neq 0} \frac{1}{m}\, e^{-imq\xi}\, \mathrm{Ei}\left(imq\xi\right) . \tag{A.19}$$

The singular behavior at $\zeta = \infty$ may be studied in the same way, or inferred from (A.6):

$$\tilde{\xi}(\zeta) \underset{\zeta\to \infty}{\sim} -\frac{q^2}{2\pi \mathrm{i}} \left(\xi \log \frac{\mathrm{i}\xi}{\pi R\Lambda\zeta} - \xi \right) - \frac{q}{2\pi} \sum_{m\neq 0} \frac{1}{m}\, e^{-imq\xi}\, \mathrm{Ei}\left(imq\xi\right) . \tag{A.20}$$

Regular complex Darboux coordinates and transition functions. The above analysis motivates the following construction of regular complex Darboux coordinates on the twistor space $\mathcal{Z}_\mathcal{S}$. First, we introduce two functions

$$\mathcal{H}^{\pm}(\xi) \equiv \frac{1}{2\pi \mathrm{i}} \sum_{m=1}^{\infty} \frac{1}{m^2} e^{\pm imq\xi}\, \mathrm{Ei}\left(\mp \mathrm{i}mq\xi\right) = \frac{1}{2\pi \mathrm{i}} \int_0^{\pm\mathrm{i}\infty} \frac{\mathrm{d}\xi'}{\xi - \xi'}\, \mathrm{Li}_2\left(e^{\pm\mathrm{i}q\xi'}\right) . \tag{A.21}$$

Moreover, we consider a four-patch covering of $\mathbb{C}P^1$ (see figure 1, left): the first patch $\mathcal{U}_+$ surrounds the north pole and extends along the contours $\ell_\pm$ down to the equator. The

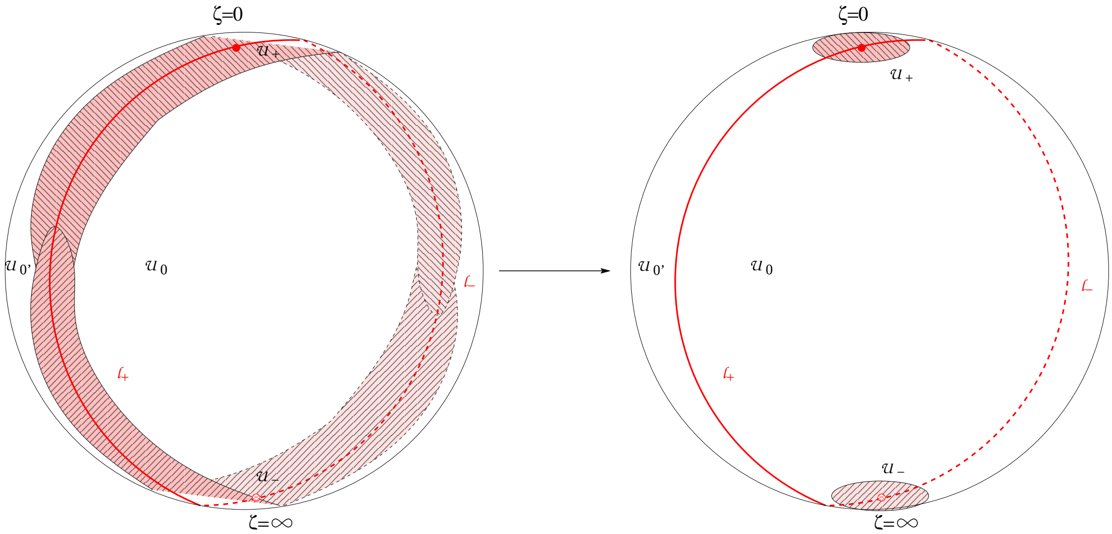

Figure 1. Two coverings of $\mathbb{C}P^1$. The covering on the left, described above (A.22), allows to reduce (A.23) to figure-eight contours around 0 and ∞. The covering on the right is obtained in the limit where the strips $\mathcal{U}_\pm$ go to zero width along the meridians $\ell_\pm$, while maintaining a non-zero size at the north and south pole.

second patch $\mathcal{U}_-$ surrounds the south pole and similarly extends halfway along $\ell_\pm$, with a non-vanishing intersection with $\mathcal{U}_+$. The rest of $\mathbb{C}P^1$ consists of two connected parts belonging to $V_>$ and $V_<$ defined above, covered by two patches $\mathcal{U}_0$ and $\mathcal{U}_{0'}$ which overlap with $\mathcal{U}_+$ and $\mathcal{U}_-$ but stay away from the contours $\ell_\pm$.

To this covering we associate the following transition functions

$$\begin{aligned} H^{[0+]} = H^{[0'+]} &= -\frac{q^2}{4\pi \mathrm{i}}\left(\xi^2 \log\frac{\mathrm{i}\zeta\xi}{\pi R\Lambda} - \frac{3}{2}\,\xi^2\right) + \mathcal{H}^+(\xi) + \mathcal{H}^-(\xi)\,, \\ H^{[0-]} = H^{[0'-]} &= \frac{q^2}{4\pi \mathrm{i}}\left(\xi^2 \log\frac{\mathrm{i}\xi}{\pi R\Lambda\zeta} - \frac{3}{2}\,\xi^2\right) - \mathcal{H}^+(\xi) - \mathcal{H}^-(\xi)\,. \end{aligned} \tag{A.22}$$

The momentum coordinates can then be obtained using the general result eq. (3.38) of [19] (adapting notations),

$$\tilde\xi^{[i]}(\zeta) = \varrho + \sum_j \oint_{C_j} \frac{\mathrm{d}\zeta'}{2\pi\mathrm{i}\,\zeta'}\,\frac{\zeta+\zeta'}{2(\zeta'-\zeta)}\,\partial_\xi H^{[0j]}(\zeta')\,, \tag{A.23}$$

which ensures that $\tilde\xi^{[0]}$ (resp. $\tilde\xi^{[0']}$) is regular in $\mathcal{U}_0$ (resp. $\mathcal{U}_{0'}$), while $\tilde\xi^{[\pm]}$ are regular in $\mathcal{U}_\pm$. Picking up the residues at $\zeta=0$ and $\zeta=\infty$ in $\mathcal{U}_+$ and $\mathcal{U}_-$, respectively, it is straightforward to check that the first term in (A.22) reproduces the weak coupling result (A.1) with $\delta\tilde\xi=0$, upon identifying $\varrho = \theta_m - \frac{q^2}{4\pi\mathrm{i}}\,\theta_e \log(a/\bar a)$.

To see that the last two terms in (A.22) reproduce $\delta\tilde\xi$, note that

$$\partial_\xi \mathcal{H}^\pm = \mp\frac{q}{2\pi}\int_0^{\pm\mathrm{i}\infty} \frac{\mathrm{d}\xi'}{\xi-\xi'}\,\log\left(1-e^{\pm\mathrm{i}q\xi'}\right) + \frac{\pi}{12\mathrm{i}\xi}\,. \tag{A.24}$$

This representation makes it apparent that $\mathcal{H}^{\pm}$ has two logarithmic cuts in the ζ-plane, extending from the north and the south poles inside $\mathcal{U}_{\pm}$, respectively, to the two zeros of ξ inside $\mathcal{U}_0 \cup \mathcal{U}_{0'}$. Due to $H^{[0+]}_{\rm inst} = -H^{[0-]}_{\rm inst}$, this situation is analogous to the one described in section 3.4 of [19]. By a similar analysis one may conclude that the corresponding contribution to $\tilde\xi^{[0]}$ (and $\tilde\xi^{[0']}$) is given by the sum of two integrals of $\partial_\xi \mathcal{H}^{\pm}$ along "figure-eight" contours encircling $\zeta=0$ and $\zeta=\infty$. Since the first term in (A.24) vanishes at these points, there are no contributions from the poles in the measure, and these integrals can be reduced to the integrals along $\ell_{\pm}$ of the discontinuity $\pm {\rm i} q \log\left(1-e^{\pm {\rm i} q\xi}\right)$ of $\partial_\xi \mathcal{H}^{\pm}$ across the cut, reproducing $\delta\tilde\xi$ in (A.3). Moreover, the last term in (A.24) does not contribute since $(\zeta\xi)^{-1}$ is regular in $\mathcal{U}_{\pm}$. Thus, the coordinates $\xi_{[0]}, \tilde\xi^{[0]}$ agree with the ones defined in [38] in the patch $\mathcal{U}_0$, and the same is true in the patch $\mathcal{U}_{0'}$.

It is perhaps useful to note that, as depicted on figure 1, the patches $\mathcal{U}_+$ and $\mathcal{U}_-$ may be shrunk to infinitesimal width along the contours $\ell_{\pm}$, while retaining a finite size around the north and south pole, respectively. However, the fact that the transition function $H^{[00']} = H^{[0+]} + H^{[+0']}$ vanishes does not imply that the coordinates $\tilde\xi$ are continuous along $\ell_{\pm}$: indeed, the transition function $H^{[0+]}$ and $H^{[+0']}$ have a discontinuity along $\ell_{\pm}$, which reproduces the shifts (A.9a) and (A.9b) in the process of analytic continuation.

A.2 Extension to QK and contact geometry

It is now straightforward to apply the above construction to D-instanton corrections to the hypermultiplet branch, since the twistor line $\tilde\xi(\zeta)$ (A.1) is essentially given by the same integral as the one appearing in (4.16). In this way, we can formulate the results of section 4 in a rigorous way, avoiding the use of open contours.

To this end, we use the same four patch covering as in the previous subsection (cf. figure 1), with the transition functions

$$
\begin{aligned}
H^{[0+]} = H^{[0'+]} &= -\frac{\rm i}{2}\, F(\xi) + \frac{1}{(2\pi)^3} {\sum_{(k_\Lambda, l^\Lambda)_+}}' n_\gamma \int_0^{-{\rm i}\infty} \frac{\Xi\, {\rm d}\Xi}{\Xi_\gamma^2 - \Xi^2}\, {\rm Li}_2\left(e^{-2\pi {\rm i}\,\Xi}\right), \\
H^{[0-]} = H^{[0'-]} &= -\frac{\rm i}{2}\, \bar F(\xi) - \frac{1}{(2\pi)^3} {\sum_{(k_\Lambda, l^\Lambda)_+}}' n_\gamma \int_0^{-{\rm i}\infty} \frac{\Xi\, {\rm d}\Xi}{\Xi_\gamma^2 - \Xi^2}\, {\rm Li}_2\left(e^{-2\pi {\rm i}\,\Xi}\right),
\end{aligned}
\tag{A.25}
$$

where

$$
\Xi_\gamma = k_\Lambda \xi^\Lambda - l^\Lambda \rho_\Lambda\,, \tag{A.26}
$$

and the only non-vanishing anomalous dimension is $c_\alpha = \chi_X/(96\pi)$. This can be shown by the same line of reasoning as below (A.24). The description based on the function (4.10) is obtained in the limit where $\mathcal{U}_+$ and $\mathcal{U}_-$ are shrunk to infinitesimal width along the contours $\ell_{\pm}$.

It would be interesting to understand the analytic structure of the twistor lines at $\mathbf{z}=0,\infty$, and thereby to extend the four patch construction described here, beyond the leading instanton approximation.

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